**Fall 2018, Math 320: Week 11 Problem Set**
**Due: Tuesday, November 13th, 2018**
**Ideals and Quotient Rings**

**Discussion problems.** The problems below should be completed in class.

(D1) *Examples of ideals.*

(a) Determine which subsets of $R = \mathbb{Z}_4$ are ideals. Do the same for $\mathbb{Z}_3$.

(b) Find all possible generators for $\langle 2 \rangle \subset \mathbb{Z}_{12}$. Do the same for $\langle 6 \rangle \subset \mathbb{Z}$.

(c) Suppose $R$ is a commutative ring and $a_1, \ldots, a_k \in R$. Fill in the blanks in the proof

$$\langle a_1, \ldots, a_k \rangle = \{b_1 a_1 + \cdots + b_k a_k : b_1, \ldots, b_k \in R\}$$

is an ideal.

*Proof.* Using Theorem 6.1, we see for any $b_1, \ldots, b_k, c_1, \ldots, c_k \in R$, we have

$$(b_1 a_1 + \cdots + b_k a_k) - (c_1 a_1 + \cdots + c_k a_k) = \underline{\hspace{3cm}} \in \langle a_1, \ldots, a_k \rangle,$$

and for any $r \in R$, we have

$$r(b_1 a_1 + \cdots + b_k a_k) = \underline{\hspace{3cm}} \in \langle a_1, \ldots, a_k \rangle,$$

as desired. $\square$

(d) Find an integer $a$ so that $\langle a \rangle = \langle 6, 10 \rangle \subset \mathbb{Z}$. Be sure to show containment both ways!

(e) Determine whether each of the following sets $I$ is an ideal in $R = \mathbb{Z}_3[x]$.

(i) $I = \{f \in R : f(0) = 0\}$.

(ii) $I = \{f \in R : f(0) = 1\}$.

(iii) $I = \{f \in R : f(1) = 0\}$.

(f) Prove or disprove: if $R$ is a ring, then the set $I$ of zero-divisors is an ideal.

(g) Prove or disprove: if $R$ is a ring, $I \subset R$ is an ideal, and $a \in I$ is a unit, then $I = \langle 1 \rangle$.

(D2) *Properties of ideals.* Assume $R$ is a commutative ring and $I \subset R$ is an ideal. In each proof, **mark in color** where ideal requirements are used.

(a) In $\mathbb{Z}_2[x]/\langle x^3+x+1 \rangle$, show that $[x^4+x^3+x^2+1] = [0]$ using the fact that $[x^3+x+1] = [0]$. With $I = \langle x^3+x+1 \rangle$, rewrite the same steps using coset notation (e.g. $(x^3+x+1)+I$).

(b) In this part, you will prove that equivalence modulo $I$ is an equivalence relation on $R$.

(i) Prove $a \equiv a \bmod I$ for every $a \in R$ (reflexivity).

(ii) Prove if $a \equiv b \bmod I$, then $b \equiv a \bmod I$ (symmetry).

(iii) Prove if $a \equiv b \bmod I$ and $b \equiv c \bmod I$, then $a \equiv c \bmod I$ (transitivity).

(c) Locate and correct the errors in the following proof that if $r+I = r'+I$ and $t+I = t'+I$, then $(rt) + I = (r't') + I$ (in other words, multiplication of cosets is well defined).

*Proof.* We must show $rt - r't' \in I$. Since $r + I = r' + I$, we have $r - r' \in I$, and since $t + I = t' + I$, we have $t - t' \in I$. As such,

$$(rt) - (rt) = rt - rt + rt - rt = (r - r)t + r(t - t) \in I,$$

as desired. $\square$

(d) State and prove a result analogous to part (c) that addition of cosets is well-defined.

(e) In order for $I$ to be an ideal, we require $ra \in I$ for every $a \in I$ and $r \in I$. If instead, we only assume $I$ is closed under multiplication, which of the above proofs breaks?

**Required problems.** As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

All rings are commutative unless otherwise stated.

(R1) Find a single generator for $I_1 = \langle 24, 34 \rangle \subset \mathbb{Z}$ (that is, find some $a \in I_1$ so that $I_1 = \langle a \rangle$). Do the same for $I_2 = \langle x^3 + 2x, x^2 + x \rangle \subset \mathbb{Q}[x]$.

(R2) Suppose $R$ is a ring. An element $r \in R$ is *nilpotent* if $r^k = 0$ for some $k \geq 1$ (that is, if **some** power of $r$ is 0). Prove that the set $N$ of all nilpotent elements of $R$ is an ideal of $R$. (Fun fact: $N$ is called the *nilradical* of $R$). Hint: you may cite old homework problems!

(R3) Suppose $R$ and $S$ are rings, and that $\phi : R \to S$ is a homomorphism. Prove that the set

$$K = \{a \in R : \phi(a) = 0_S\}$$

is an ideal of $R$.

(R4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If $I, J \subset R$ are ideals, then $I \cap J$ is an ideal.

(b) If $I, J \subset R$ are ideals, then $I \cup J$ is an ideal.

(c) If $I, J \subset R$ are ideals, then $I + J = \{a + b : a \in I, b \in J\}$ is an ideal.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Consider the ring

$$R = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\} \subset \mathbb{R}$$

(you may assume that $R$ is a subring of $\mathbb{R}$). Locate an ideal $I \subset \mathbb{Q}[x]$ so that $\mathbb{Q}[x]/I \cong R$.