Fall 2018, Math 320: Week 12 Problem Set
Due: Tuesday, November 20th, 2018
More Ideals and Quotient Rings

Discussion problems. The problems below should be completed in class.

(D1) *Using the first isomorphism theorem.* The goal of this problem is to use the first isomorphism theorem (stated below) to prove $\mathbb{Z}[x]/\langle x\rangle \cong \mathbb{Z}$.

Theorem. *If $\varphi : R \to S$ is a surjective homomorphism and $I = \ker(\varphi)$, then $R/I \cong S$.*

(a) Define $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ by $\varphi(f(x)) = f(0)$ (each polynomial maps to its constant term). Find $\varphi(x + 2)$, $\varphi(x^2 + 2x + 7)$, and $\varphi(x^5 + 5x^2)$.

(b) Complete the following proof that φ is surjective in **two different ways**.

Proof. For each $a \in \mathbb{Z}$, we have $\varphi(\_\_\_\_\_\_\_) = a$, so φ is surjective. □

(c) Let $I = \ker(\varphi)$. Locate the error in the following proof that $I = \langle x\rangle$.

Proof. If $f(x) \in I$, then $f(0) = 0$, so by the root theorem, we have $f(x) = xg(x)$ for some $g(x) \in \mathbb{Z}[x]$. As such, $f(x) \in \langle x\rangle$. This proves I and $\langle x\rangle$ are identical sets. □

(d) Prove φ is a homomorphism.

(e) Use the first isomorphism theorem to conclude $\mathbb{Z}[x]/\langle x\rangle \cong \mathbb{Z}$. Are equivalence classes used in any of the above proofs?

(f) Prove $\mathbb{Z}[x]/\langle x\rangle \cong \mathbb{Z}$ directly, by showing $\phi : \mathbb{Z}[x]/\langle x\rangle \to \mathbb{Z}$ given by $(f(x) + I) \mapsto f(0)$ is an isomorphism. Hint: do we need to show ϕ is well-defined?

(D2) *Proving the first isomorphism theorem.* The goal of this problem is to prove the first isomorphism theorem (above). Let R, S, I, and φ be as in the theorem statement.

(a) Define $\phi : R/I \to S$ by $(r + I) \mapsto \varphi(r)$. Write ϕ and φ together on top of the board. Idenfity which rings r, $r + I$, $\varphi(r)$, and $\phi(r + I)$ each live in.

(b) Justify each claimed "=" in the following proof that ϕ is well-defined (use color!).

Proof. Suppose $r + I = r' + I$. We must show $\varphi(r) = \varphi(r')$. Since $r + I = r' + I$ implies $r - r' \in I$, we have $0 = \varphi(r - r') = \varphi(r) - \varphi(r')$, meaning $\varphi(r) = \varphi(r')$. □

(c) Locate the error in the following proof that ϕ is surjective.

Proof. Suppose $t \in S$. We must show t lies in the image of ϕ. Since φ is surjective, $\varphi(r) = t$ for some $r \in R$. As such, $\phi(r) = \varphi(r) = t$. □

(d) **Without looking below**, use Theorem 6.11 to prove ϕ is injective. Once you finish, compare your proof to the one below. Lastly, identify which ring each "0" lives in.

Proof. Using Theorem 6.11, we will prove $\ker(\phi) = \{0 + I\}$. Suppose $r + I \in \ker(\phi)$. This means $\phi(r + I) = 0$, so $\varphi(r) = 0$, and $r \in \ker(\varphi) = I$. As such, $r + I = 0 + I$. □

(e) Prove ϕ is a homomorphism. Conclude the first isomorphism theorem holds.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

All rings are commutative unless otherwise stated.

(R1) Prove that the map $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_4$ given by $[a]_{12} \mapsto [a]_4$ is a surjective ring homomorphism (be sure to show it is well-defined!). Find the kernel of φ.

(R2) Find an element $r \in \mathbb{Q}[x]/\langle x^4 + x^2 \rangle$ other than $r = [0]$ and $r = [1]$ with the property $r^2 = r$ (we say r is *idempotent*).

(R3) Suppose R is a ring. Use the first isomorphism theorem to prove $R/\langle 0 \rangle \cong R$.

(R4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

- (a) If R is an integral domain and $I \subset R$ is an ideal, then R/I is an integral domain.
- (b) If F is a field, then the only ideals $I \subset F$ are $I = \{0\}$ and $I = F$.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Suppose R is a ring and $I, J \subset R$ are ideals. Prove that the set $L = \{(a+I) \in R/I : a \in J\}$ is an ideal of R/I.

(S2) Suppose $I, J \subset R$ are ideals, and let $\varphi : R \to (R/I) \times (R/J)$ given by $a \mapsto (a + I, a + J)$. Prove that φ is a homomorphism, and express $\ker(\varphi)$ in terms of I and J.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose R and S are rings, and $\varphi : R \to S$ is a surjective homomorphism with $I = \ker \varphi$. Prove that there is a bijection

$$\{\text{ideals } J \subset R \text{ with } I \subset J\} \longrightarrow \{\text{ideals of } S\}$$

given by $J \mapsto \{\varphi(a) : a \in J\}$ (we say φ *induces* this bijection).