Fall 2018, Math 320: Week 14 Problem Set
Due: Tuesday, December 4th, 2018
Introduction to Groups

Discussion problems. The problems below should be completed in class.

(D1) *Checking group axioms.* Determine whether each of the following sets G forms a group under the given operation $*$.

(a) $G = \mathbb{Z}$; $a * b = a \cdot b$.

(b) $G = \mathbb{Z}$; $a * b = a - b$.

(c) G is the set of nonzero rational numbers; $a * b = a/b$.

(d) $G = \mathbb{Z}_{\geq 0}$; $a * b = a + b$.

(e) $G = \mathbb{Z}_{\geq 1}$; $a * b = ab$.

(f) $G = \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).

(g) $G = \{1, 3, 7, 9\} \subset \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).

(h) $G = \{1, 3, 7, 9\} \subset \mathbb{Z}_9$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_9$).

(i) $G = \{1, 2, 4, 5, 7, 8\} \subset \mathbb{Z}_9$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_9$).

(j) $G = \left\{x, 1 - x, \frac{1}{x}, \frac{1}{1-x}, \frac{x}{1-x}, \frac{1-x}{x}\right\}$; $f(x) * g(x) = f(g(x))$.

(k) G is the set of functions $\mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ with $a, b \in \mathbb{R}$ and $a \neq 0$ (that is, $f(x) = 2x + 3$ and $f(x) = 5x$ lie in G, but not $f(x) = x^2 + 2$ and $f(x) = 0$); $f(x) * g(x) = f(g(x))$.

(l) $G = \mathbb{R} \times \mathbb{R}$; $(a, b) * (c, d) = (ac, bd)$.

(m) $G = \mathbb{R}^* \times \mathbb{R}$ where $\mathbb{R}^*$ denotes the set of nonzero real numbers; $(a, b) * (c, d) = (ac, bc + d)$.

(D2) *Graph automorphisms.* Recall that an *automorphism* of a graph H is a permutation σ of the vertices of H such that $\sigma(a)$ and $\sigma(b)$ are connected whenever a and b are connected.

For each of the following graphs H, find $\mathbb{A}(H)$, the group of all automorphisms of H. For each, identify which "key" group from Tuesday's lecture $\mathbb{A}(H)$ is isomorphic to.

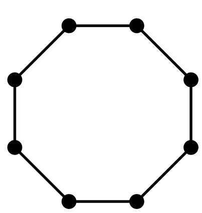
(a) The cycle graph $G = C_n$ for $n \geq 3$.

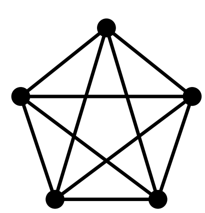
(b) The complete graph $G = K_n$ for $n \geq 2$.

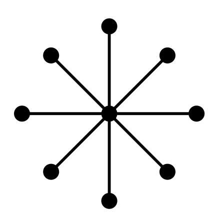
(c) The star graph $G = T_n$ for $n \geq 3$.

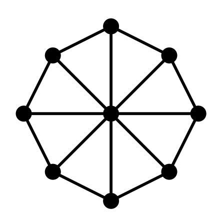
(d) The wheel graph $G = W_n$ for $n \geq 3$.

Use the following graphs as a guide for this problem.


Cycle graph C_8


Complete graph K_5


Star graph T_8


Wheel graph W_8

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Consider the dihedral group $D_3 = \{e, r, r^2, f_1, f_2, f_3\}$, the symmetry group of the triangle with vertices labeled $1, 2, 3$ when read clockwise, where r denotes a $120°$ clockwise rotation and each f_i denotes the flip leaving vertex i fixed.

(a) Write each element of D_3 in permutation notation, e.g.

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

(b) Write out the operation table for D_3.

(R2) Each of the following sets of 2×2 real matrices does **not** form a group under matrix multiplication. Determine why (you may assume matrix multiplication is associative).

(a) $M = M(\mathbb{R})$ (that is, the set of **all** 2×2 matrices with real entries).

(b) $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ with } a \neq 0 \text{ and } b \neq 0 \right\}$.

(R3) Determine whether each of the following sets G form a group under the given operation $*$. If yes, prove they form a group. If no, give a specific example demonstrating that one of the axioms is violated.

(a) G is the set of nonzero real numbers; $a * b = |a| \cdot b$.

(b) $G = \mathbb{R}$; $a * b = a + b + 3$.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

For **this week only**, you are **not** required to give a complete argument. Instead, simply state your answer (a short expression for (S1), a picture for (S2)) and give a one sentence explanation.

(S1) Suppose H_1 and H_2 are graphs, and suppose H is the disjoint union of H_1 and H_2. Express $\mathbb{A}(H)$ in terms of $\mathbb{A}(H_1)$ and $\mathbb{A}(H_2)$.

(S2) Identify a graph H whose automorphism group $\mathbb{A}(H)$ is isomorphic to $(\mathbb{Z}_5, +)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose $(G, *)$ is a group, where $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $*$ is an operation satisfying

(i) $a * b \leq a + b$ for every $a, b \in G$, and

(ii) $a * a = 0$ for every $a \in G$.

Write out the operation table for G.