Fall 2018, Math 320: Week 15 Problem Set
Due: Tuesday, December 11th, 2018
Permutation Groups

Discussion problems. The problems below should be completed in class.

(D1) *Working with permutations.* Consider the following permutations.

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,5\,2\,4\,1 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,1\,6\,3\,8\,2\,5\,7 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 5\,2\,8\,6\,1\,4\,7\,3 \end{pmatrix}$$

(a) Write each of the above permutations as a product of disjoint cycles.

(b) Write each of the above permutations as a product of 2-cycles. Is there more than one way to do this?

(c) Determine which of the above permutations are even, and which are odd.

(d) Is it possible to write σ or τ as a product of disjoint 2-cycles?

(e) Write σ^{-1}, τ^{-1}, and α^{-1} as a products of disjoint cycles, and as products of 2-cycles. Hint: you have already written σ, τ, and α in these forms!

(f) Determine whether σ^{91} is even or odd.

(g) Determine the smallest integer $m > 0$ for which α^m equals the identity permutation. Do the same for σ and τ.

(D2) *Group elements as permutations.* The goal of this problem is to see the following in action.

Theorem. *Every finite group G with $|G| = n$ is isomorphic to a subgroup of S_n.*

(a) Draw the triangle with vertices labeled $1, 2, 3$. Write each element of D_3 as a permutation of the vertex labels.

(b) Conclude that D_3 is isomorphic to a subgroup of S_3.

(c) Which elements of D_3 correspond to odd permutations in S_3?

(d) The goal of the next few parts is to identify a subgroup of S_6 isomorphic to D_3. Choose a labeling of the elements of D_3 as $a_1, a_2, \ldots, a_6$ (you may do this in any way you wish).

(e) Let $r \in D_3$ denote clockwise rotation by 120°. Define a permutation $\sigma \in S_6$ given by $\sigma(i) = k$ where $a_k = a_i r$. Verify that σ is indeed a permutation by writing it in permutation notation.

(f) Repeat the previous part for each $a \in D_3$ (that is, define a permutation $\sigma_a \in S_6$ given by $\sigma(i) = k$ where $a_k = a_i a$). Write down all 6 resulting permutations (you may want to "divide and conquer" amongst your groupmates to save time!).

(g) Verify in three examples that for any $a, b \in D_3$, the permutation corresponding to ab equals the product of the permutations corresponding to a and b.

(h) Which elements of D_3 correspond to odd permutations in S_6? Does this match part (c)?

(i) Using the ideas above, find a subgroup of S_5 isomorphic to $\mathbb{Z}_5$.

(j) Using the ideas above, find a subgroup of S_4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Write σ and τ below as products of disjoint cycles, and as products of 2-cycles. Do the same for σ^{-1} and τ^{-1}.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 7 & 2 & 8 & 5 & 1 & 6 \end{pmatrix}$$

(R2) Let B_n denote the set of odd permutations in S_n. Define $f : A_n \to B_n$ by $f(\sigma) = \sigma \cdot (12)$.

- (a) Prove f is injective.
- (b) Prove f is surjective.
- (c) Prove that $|A_n| = n!/2$ (hint: what do parts (a) and (b) imply about $|A_n|$ and $|B_n|$?).

(R3) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

- (a) The set B_n of odd permutations in S_n is a subgroup of S_n.
- (b) There is a subgroup of S_{36} isomorphic to D_{18}.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove that (12) and $(123 \cdots n)$ generate S_n.