Fall 2019, Math 579: Problem Set 1
Due: Thursday, September 5th, 2019
Induction and the Pigeon-Hole Principle

Discussion problems. The problems below should be worked on in class.

(D1) *Strong induction.* Fix a statement $P(n)$ dependent on n, and suppose that:

- $P(1)$ holds (the *base case*); and
- if $P(k)$ holds for all $1 \leq k \leq n$, then $P(n+1)$ holds (the *inductive step*).

We can conclude that $P(n)$ holds for all n. This technique is called *strong induction* on n. (How does this differ from usual (*weak*) induction?)

Prove the following results using induction. For each, indicate whether your proof uses strong induction or weak induction.

(a) Define $a_0 = 0$ and
$$a_n = a_0 + \cdots + a_{n-1} + n$$
for all $n \geq 1$. Prove that $a_n = 2^n - 1$ for $n \geq 0$.

(b) Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \geq 1$.

(D2) *The generalized pigeon-hole principle.* Consider the following generalization of the PHP.

Theorem. *Fix positive integers $n, m, r > 0$, and suppose $n > rm$. If n pigeons are placed into m boxes, then some box contains at least $r + 1$ pigeons.*

(a) Suppose 9 integers are selected at random. Prove that at least 5 have the same parity (even or odd). What if only 8 integers are selected?

(b) Prove the generalized pigeon-hole principle using induction on r. Be sure to carefully state your assumptions for each step! You may use the standard pigeon-hole principle in both the base case and inductive step.

(c) Does your argument for part (b) use strong induction?

(D3) *Using the pigeon-hole principle.*

(a) Solve each of the following problems using the pigeon-hole principle. Be sure to specify the version of the pigeon-hole principle used.

(i) If 10 points are chosen inside of a unit square, then there are two points with a distance at most 0.5 apart.

(ii) If 10 points are chosen inside of a unit square, then three points can be covered by a disk of radius 0.5.

(b) Locate an error in the following proof that if $a_1 + \cdots + a_{100} = 0$, there are at least 99 pairs a_i, a_j whose sum is non-negative. You are **not** required to correct the error!

Proof. Consider the following collections of pairs.

- $a_1 + a_2, \quad a_3 + a_4, \quad \cdots, \quad a_{99} + a_{100}.$
- $a_1 + a_3, \quad a_5 + a_7, \quad \cdots, \quad a_{97} + a_{99}, \quad a_2 + a_4, \quad a_6 + a_8, \quad \cdots, \quad a_{98} + a_{100}.$
- $a_1 + a_4, \quad \cdots, \quad a_{97} + a_{100}, \quad a_3 + a_6, \quad \cdots, \quad a_{99} + a_2, \quad a_5 + a_8, \quad \cdots, \quad a_{97} + a_{99}.$

In each item above, each a_i appears exactly once, so the total sum is 0, so the pairs cannot all be negative. Moreover, all pairs have difference 1 in the first item, 2 in the second item, etc. Continuing in this way, we obtain 1 non-negative pair for each difference $1, 2, \ldots, 99$, as desired. □

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Define the sequence g_n as follows: $g_0 = 0$, $g_1 = 1$, and $g_n = g_{n-1} + g_{n-2}$. Use induction to prove that

$$g_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

holds for all $n \geq 0$. Does your proof use strong induction?

(H2) Suppose 400 balls are distributed into 200 boxes in such a way that no box contains more than 200 balls, and each box contains at least one ball. Then there are some boxes which together contain exactly 200 balls.

(H3) Prove that among 1002 positive integers, there are always two integers whose sum or difference is a multiple of 2000.

(H4) Suppose every point in $\mathbb{N}^2$ is colored using one of 8 colors.

(a) Prove that there exists a rectangle whose vertices are monochromatic.

(b) Suppose $\mathbb{N}^2$ is colored using one of r colors, where $r > 0$. For which values of r does part (a) still hold?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove the round robin tournament theorem: for $n \geq 2$, there exists an n-player tournament with $n-1$ rounds if n is even (and n rounds if n is odd) in such a way that any 2 players compete against each other exactly once.