Fall 2019, Math 579: Problem Set 2
Due: Thursday, September 12th, 2019
Elementary Counting Methods

Discussion problems. The problems below should be worked on in class.

(D1) *Counting arguments.* Each of the following proofs has an error. Copy each **full proof** onto the board, then locate and correct the error **in a different color**.

(a) Find the number of ways to order 112345 so that the 1's are not adjacent.

Proof. First, there are 6! ways to order 6 distinct symbols, but since '1' occurs twice, we must divide by 2!. Now, if we first count the orderings where the 1's **are** adjacent, we can treat '11' as a single symbol, yielding 5! orderings. As such, we get

$$\frac{6!}{2! \cdot 5!}$$

total ways when the 1's **are not** adjacent. □

(b) Find the number of ways to choose 3 appetizers and 2 desserts from a menu with 5 appetizers and 6 desserts.

Proof. For the appetizers: there are $\binom{5}{3}$ ways to choose 3 out of the 5 appetizers. For dessert: there are 6 choices for the first dessert and 5 choices for the second dessert, yielding $6 \cdot 5$ possibilities. As such, we obtain

$$\binom{5}{3} \cdot 6 \cdot 5$$

ways to place the full order. □

(D2) *Poker hands.* Suppose you have a 52 card deck, with 4 suits (labeled spades, clubs, hearts, diamonds) and 13 ranks (labeled 2 through 10, Jack, Queen, King, and Ace). Each card has one rank and one suit, and no two cards are identical.

(a) Determine the total number of possible 5-card hands.

(b) Determine the number of poker hands that have each ranking:

- royal flush (ranks 10-Ace, all the same suit);
- straight flush (sequential ranks, all the same suit);
- 4 of a kind (4 cards have the same rank);
- full house (3 cards have one rank, 2 cards have another rank);
- flush (all cards have the same suit);
- straight (cards have sequential ranks);
- 3 of a kind (3 cards have the same rank);
- 2 pair (2 cards have one rank, 2 others have the same rank);
- 1 pair (2 cards have one rank); and
- high card (none of the above).

Each 5-card hand should fall under exactly one name (e.g., a 2 pair is *not* a pair).

(c) Verify that the sum of all of your answers from part (b) yields the same number as in part (a), and that hands with higher rankings occur less frequently.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose $b_1 + \cdots + b_m \le n$. Prove that $b_1! \cdots b_m! \le n!$.

(H2) Determine the number of ways to place n non-attacking rooks on an $n \times n$ chess board (a rook can move in a straight line up, down, left, and right).

(H3) Prove that exactly half of the subsets of $[n]$ have an even number of elements.

(H4) In how many ways can we select two subsets $C, D \subset [n]$ such that $C \cap D = \emptyset$?

(H5) Let P denote a convex n-sided polygon in which no 3 diagonals intersect in a single point. How many intersection points do the diagonals of P have?

(H6) Suppose you have an $n \times n$ matrix in which each element of $[n^2]$ appears exactly once, and the sum of every row and column is exactly r. Find all possible values of r.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) How many non-attacking queens can be placed on an $n \times n$ chess board (queens can move in a straight line up, down, left, right, and diagonal)?

(C2) How many $n \times n$ square matrices are there whose entries are 0's and 1's and in which every row and column has an even sum?