Fall 2019, Math 579: Problem Set 3
Due: Thursday, September 19th, 2019
Binomial Theorem and Inclusion-Exclusion

Discussion problems. The problems below should be worked on in class.

(D1) *Combinatorial proofs and the binomial theorem.*

(a) Fill in the blanks in the following **combinatorial** proof that for any $n \ge 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Proof. We will count the number of ways to choose subsets $A, B \subseteq [n]$ with $A \subseteq B$.

Right side: for each $i \in [n]$, either $i \in A$, $i \in B \setminus A$, or ____. This yields 3^n possibilities.

Left side: if we let $k = |B|$, then for each $k = 0, 1, \ldots, n$, there are ____ choices for B, and ____ ways to choose a subset $A \subseteq B$. This yields

$$2^0 \binom{n}{0} + 2^1 \binom{n}{1} + \cdots + 2^n \binom{n}{n}$$

possibilities. We conclude the left side must equal the right side. $\square$

(b) Give an **algebraic** proof (i.e., using the binomial theorem) of the identity in part (a).

(c) Give a **combinatorial** proof that whenever $k \ge 0$ and $k + 3 \le n$,

$$\binom{n}{3}\binom{n-3}{k} = \binom{n}{k}\binom{n-k}{3}.$$

(d) Consider the following identity, valid whenever $n \ge 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}.$$

(i) Give a **combinatorial** proof of this identity.
(ii) Give an **algebraic** proof of this identity. Hint: consult Preliminary Problem (P1).
(iii) Give a proof by induction. Hint: be **very** careful with your indices!

(D2) *Applications of Inclusion-Exclusion.* Recall the Sieve formula from Tuesday:

$$|A_1 \cup \cdots \cup A_n| = \sum_{\substack{T \subseteq [n] \\ T \ne \emptyset}} (-1)^{|T|+1} \left| \bigcap_{j \in T} A_j \right|.$$

(a) How many positive integers less than 100 are relatively prime to either 2 or 3?

(b) How many 3-digit positive integers are divisible by 6, 7, or 8? Clearly label A_1, A_2, A_3.

(c) In part (b), for which sets T in the Sieve formula does 24 appear in $\bigcap_{j \in T} A_j$?

(d) In the general Sieve formula at the start of this problem, suppose $a \in A_1 \cup \cdots \cup A_n$ and that a lies in exactly m of the sets $A_1, \ldots, A_n$. Explain why, on the right hand side of the Sieve formula, a is counted the following number of times:

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \cdots + (-1)^{m-1}\binom{m}{m}.$$

(e) Give an **algebraic** proof that the expression above equals 1 for any m.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the coefficient of $x^{11}y^7$ in the expansion of $(x+y)^{18} + x^3(x-y)^{15}$.

Hint: **don't expand**!!! This is what the binomial theorem is for!!!

(H2) Use induction on n to prove that for all $n \geq 1$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: use the identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

in your inductive step.

(H3) Give an **algebraic** proof (that is, using the binomial theorem) that for all $n \geq 1$,

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} = \frac{1}{n+1}.$$

(H4) Give a **combinatorial** proof that for all $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

(H5) How many ways can we arrange the characters $3, 3, 4, 4, 5, 6, 7$ so that no two consecutive digits are identical?

(H6) Find $\phi(210)$, where $\phi(n)$ denote the number of elements of $[n]$ relatively prime to n.

Hint: $210 = 2 \cdot 3 \cdot 5 \cdot 7$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give a combinatorial proof of the identity

$$(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r}.$$

(C2) Let $\phi(n)$ denote the number of elements of $[n]$ relatively prime to n. Find a formula for $\phi(p_1^{a_1} \cdots p_k^{a_k})$, where $p_1, \ldots, p_k$ are distinct primes and $a_1, \ldots, a_k \geq 1$. Hint: first prove this for the case $k = 1$.