Fall 2019, Math 579: Problem Set 4
Due: Tuesday, September 24th, 2019
Partitions

Discussion problems. The problems below should be worked on in class.

(D1) *Bell numbers.* Recall that $S(n, k)$ denotes the number of set partitions of $[n]$ into exactly k blocks (called the *Stirling numbers of the 2nd kind*). Let $B(n)$ denote the total number of set partitions of $[n]$, that is, $B(n) = \sum_{k=1}^{n} S(n, k)$ (called the n-th *Bell number*).

(a) Find $B(1)$, $B(2)$, $B(3)$, and $B(4)$. Which of these did you find in Problem (P1)?

(b) Find $B(5)$ using the identity

$$B(n+1) = 1 + \sum_{k=1}^{n} \binom{n}{k} B(k).$$

(c) Give a combinatorial proof of the identity in part (b).

(d) Give a combinatorial proof that for all $n \geq 1$,

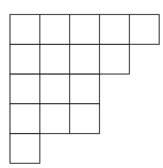
$$B(n+1) - B(n) = \sum_{k=1}^{n} kS(n, k).$$

(D2) *Ferres diagrams.* Recall that $p(n)$ counts the number of integer partitions of n, and $p_k(n)$ counts the number of partitions into exactly k parts.

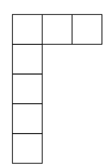
(a) Find $p(6)$ and $p_3(6)$, and list the corresponding partitions.

(b) Prove that for $n \geq 2$, the number of partitions of n with no 1's is given by $p(n)-p(n-1)$.

(c) Given a partition of n, the associated *Ferrer diagram* or *Young diagram* is a drawing of n boxes that are "upper-left justified" where the number of boxes in the i'th row equals the size of the i'th part. See below for examples.



$5 + 4 + 3 + 3 + 1$



$3 + 1 + 1 + 1 + 1$

Find all partitions of 7 into 3 parts, and write the Ferrer diagram for each partition.

(d) Given a Ferrer diagram of a partition, what is the number of rows equal to? What is the number of columns equal to?

(e) Using part (d), find a bijection between the partitions of n into at most k parts and the partitions of n into parts of size at most k.

(f) Is it true that $p_k(n)$ equals the number of partitions of n into parts of size exactly k?

(g) The partition obtained by reflecting a Ferres diagram about its diagonal (which you likely used in the preceeding two problems) is called the *conjugate* partition. Explain why the conjugate of a partition in which every part appears an even number of times is a partition whose parts are all even. Is the converse true?

(h) Prove that the number of partitions of n into at most k parts equals $p_k(n + k)$.

(i) A partition is *self-conjugate* if it equals its conjugate. Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the number of compositions of 5. Find the number of compositions of 10 into all even parts. How are these related?

(H2) Use the recurrence identity $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$ from class to complete the next 3 rows of the triangle of Stirling numbers of the 2nd kind (the analog of Pascal's triangle where the entry in row n and position k is $S(n,k)$).

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 7 & & 6 & & 1 \end{array}$$

(H3) Find a closed formula for $S(n, n-2)$ for all $n \geq 3$.

(H4) Let $F(n)$ denote the number of set partitions of $[n]$ with no singleton blocks. Prove that $B(n) = F(n) + F(n+1)$.

(H5) A partition is *self-conjugate* if it equals its conjugate. Prove that the number of self-conjugate partitions of n has the same parity as $p(n)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let $F(n)$ denote the number of all partitions of $[n]$ with no singleton blocks. Find a recursive formula for $F(n+1)$ in terms of $F(k)$ for $k \leq n$, and give a combinatorial proof of its correctness.