Fall 2019, Math 579: Problem Set 5
Due: Thursday, October 10th, 2019
Introduction to Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Graph automorphisms.* An *automorphism* of a graph G is a bijection $f : V(G) \to V(G)$ such that $(v, w) \in E$ precisely when $f(v) = f(w)$.

(a) Draw the cycle graph C_4, and label the vertices with $1, 2, 3, 4$ in a clockwise fashion. Write all 8 automorphisms of C_4. As an example, one automorphism f is given by

$$f(1) = 2, \qquad f(2) = 3, \qquad f(3) = 4, \qquad \text{and} \qquad f(4) = 1.$$

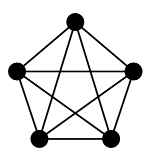
(b) Complete the following proof that for each $n \geq 3$, the n-vertex cycle graph $G = C_n$ has exactly $2n$ automorphisms. Draw an accompanying "graph fragment" picture too.

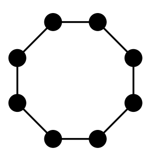
Proof. Label the vertices of G by $1, 2, \ldots, n$ in a clockwise fashion. For convenience, if we refer to vertex $n + 1$, we mean vertex 1, and similarly for vertex $n + 2$, etc. Suppose $f : V(G) \to V(G)$ is an automorphism of G, and let $v = f(1)$. We must have $f(2) = v + 1$ or ____ since $(1, 2) \in E(G)$ and these are the only vertices connected to v. If $f(2) = v + 1$, then $f(3) =$ ____, $f(4) =$ ____, and so on since __________. Similarly, if $f(2) =$ ____, the remaining values of f must again follow in cyclic order. This means f is determined by choosing the value $f(1)$ (___ possibilities) and then an adjacent value for $f(2)$ (___ possibilities), yielding $2n$ total automorphisms. □

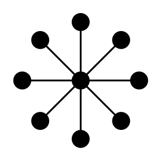
(c) Prove that if $n \geq 2$ and $G = K_n$ is the complete graph with vertex set $[n]$, then **every** bijection $V(G) \to V(G)$ is an automorphism.

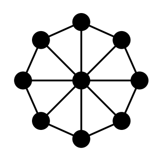
(d) Find (with proof!) all automorphisms of the star graph $G = S_n$ with $n \geq 2$ appendages.

(e) Find (with proof!) all automorphisms of the wheel graph $G = W_n$ for $n \geq 3$ spokes.


Complete graph K_5

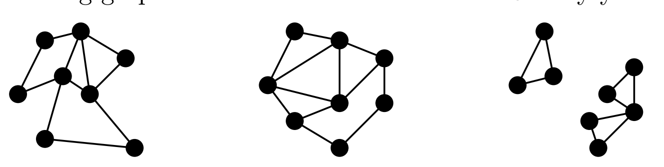

Circle graph C_8


Star graph S_8


Wheel graph W_8

(D2) *Eulerian walks.* Fix a graph $G = (V, E)$. A walk on G is *Eulerian* if every edge in G is used exactly once.

(a) Which of the following graphs have *closed* Eulerian walk? Justify your answers.



(b) Have each member of your group draw a graph with at least 8 vertices and 15 edges. Together, determine which of these graphs has a closed Eulerian walk.

(c) Find (and prove!) a criterion for when a simple graph G has a closed Eulerian walk.

(d) Which simple graphs have a (not necessarily closed) Eulerian walk? Try a few first!

(e) Make each of the above graphs *directed* by manually adding a direction to each edge. Now, which graphs have a closed Eulerian walk? Do this several times for each graph.

(f) If G is directed, what conditions must it satisfy in order to adapt your proof in part (c)?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that if there is a walk from a vertex v_1 to a vertex v_2 in a simple graph G, then there is a path from v_1 to v_2.

(H2) Is there a disconnected simple graph on 7 vertices such that every vertex has degree at least 3?

(H3) Suppose G is a k-regular graph (that is, a simple graph in which every vertex has degree exactly k). Prove that G has a cycle of length at least $k + 1$.

(H4) Find all non-isomorphic simple graphs on four vertices. Be sure to prove you found them all!

(H5) Find a simple graph G that has no nontrivial automorphisms (that is, where the only automorphism is the identity map).

(H6) A *Hamiltonian cycle* is a cycle which visits every vertex exactly once. How many distinct Hamiltonian cycles does the complete graph K_n have?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) If G is a simple graph with n vertices and no Hamiltonian cycles, then what is the maximum number of edges G can have? (Your answer should depend on n.)