Fall 2019, Math 579: Problem Set 6
Due: Thursday, October 17th, 2019
Trees

Discussion problems. The problems below should be worked on in class.

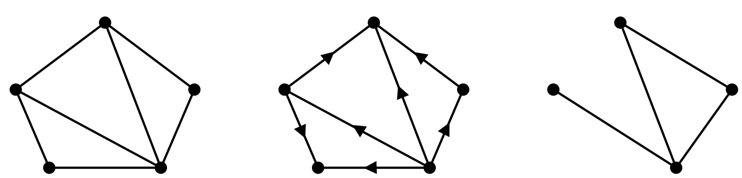
(D1) *Counting walks of fixed length.* Let G be a graph G and let A be its adjacency matrix.

- (a) Find the adjacency matrix of K_5, the complete graph on 5 vertices. Verify that the entry $(A^2)_{3,4}$ equals the number of walks from vertex 3 to vertex 4 of length 2.
- (b) Suppose B and C are matrices in which B_{ij} equals the number of walks from vertex i to vertex j of length 5, and C_{ij} equals the number of walks from vertex i to vertex j of length 7. Give a combinatorial proof that $(BC)_{3,4}$ equals the number of walks from vertex 3 to vertex 4 of length 12.
- (c) Justify why $(A^k)_{ij}$ equals the number of walks from vertex i to vertex j of length k.
- (d) Recall that the determinent of an upper-triangular matrix is the product of the diagonal entries. Find the determinent of the matrix in Preliminary Problem (P3) by first adding a multiple of one row to another and then using this fact.
- (e) Use the matrix-tree theorem from class to find the number of spanning trees of K_5. Hint: start by adding every row to the first row.

(D2) *Counting spanning trees.* Fix a directed graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The *incidence matrix* of G is the $n \times m$ matrix A defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j; \\ -1 & \text{if } v_i \text{ is the tail of } e_j; \\ 0 & \text{otherwise} \end{cases}$$

- (a) Find all spanning trees in the undirected graph depicted on the left below.



- (b) Find the incidence matrix A of the directed graph depicted in the middle below.
- (c) Consider the matrix A_0 obtained by omitting the last row of A. Compute the determinant of several 4×4 submatrices of A_0 (divide the work on this!).
- (d) Notice that the value of each determinant in part (c) is either 0 or 1. What do you notice about the edges corresponding to the columns when this value is 1?
- (e) Fix an arbitrary directed graph G with incidence matrix A, and let A_0 denote the matrix obtained by removing the last row of A. Consier the Binet-Cauchy formula

$$\det(A_0 A_0^T) = \textstyle\sum_B (\det B)^2_,$$

where the sum ranges over all $(n-1) \times (n-1)$ submatrices B of A_0. Use this and part (d) to show $\det(A_0 A_0^T)$ equals the number of (undirected) spanning trees of G.

- (f) Fix an undirected graph G with vertices $v_1, \ldots, v_n$, and consider the directed graph G' obtained from G by replacing each undirected edge of G with two directed edges, one in each direction. Let A denote the incidence matrix for G'. Compute AA^T for the undirected graph G depicted on the right in part (a) above.
- (g) Now find the adjacency matrix for G. What do you notice about these two matrices? Use this to prove the matrix tree theorem.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find all non-isomorphic trees on 6 vertices.

(H2) How many different trees are there on $[n]$ whose vertices have degree at most 2? How many such trees are there up to isomorphism?

(H3) Prove that in any tree G, any two longest paths cross each other. Is the same true if G is connected but not necessarily a tree?

(H4) Suppose G is a tree, and no vertex of G has degree more than 3. Prove that the number of vertices with degree 1 is two more than the number of vertices with degree 3.

(H5) Find the number of spanning trees of the circle graph C_n. Verify your answer using the matrix tree theorem.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Find the number of spanning trees of the wheel graph W_n.