Fall 2019, Math 579: Problem Set 7
Due: Thursday, October 24th, 2019
Matching and Bipartite Graphs

(D1) *Chromatic polynomials.* Fix a graph G with $n = |V(G)|$. The *chromatic function* of G is

$$\chi_G(k) = \# \text{ proper colorings of } G \text{ with } k \text{ colors.}$$

(a) Find the chromatic function of each of the following graphs.

(i) The competely disconnected graph D_n with n vertices and no edges.

(ii) The complete graph K_n.

(iii) The path graph P_n.

What do you notice about the "type" of each of these functions?

(b) Let c_i denote the number of ways to properly color G using **exactly** i colors. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \sum_{i=1}^{n} \binom{k}{i} c_i = \binom{k}{1} c_1 + \binom{k}{2} c_2 + \cdots + \binom{k}{n} c_n$$

for all $k \geq 1$. Use this to find $\chi_G(k)$ for $G = K_3$, and verify it matches part (a).

(c) Give a combinatorial proof of the identity in part (b).

(d) Using the fact that $\binom{k}{i} = \frac{1}{i!} k(k-1)\cdots(k-i+1)$ is a polynomial in k of degree i, deduce from part (b) that $\chi_G(k)$ is a polynomial in k of degree n.

(e) Fix an edge $e \in E(G)$. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \chi_{G \setminus e}(k) - \chi_{G/e}(k).$$

Using **only** this equality and part (a)(i), find the chromatic polynomial of $G = K_3$.

(f) Use the equality in part (e) to give an inductive proof that $\chi_G(k)$ is a polynomial. Does this proof guarantee that the coefficients of $\chi_G(k)$ are all integers?

Hint: use part (a)(i) as your base case.

(g) Give a combinatorial proof of the equality in part (e).

(D2) *Hall's Marriage Theorem.* Given a graph G and $T \subset V(G)$, let $N(T)$ denote the set of neighbors of vertices in T. In this problem, you will prove the following result.

Theorem. *Fix a bipartite graph $G = (X, Y)$. There exists a perfect matching of X into Y if and only if $|T| \leq |N(T)|$ for all $T \subset X$.*

(a) First, verify Hall's Marriage Theorem holds in the special cases when $|X| = 1$, $|X| = 2$, $|Y| = 1$ and $|Y| = 2$.

(b) Prove the forward direction of Hall's Marriage Theorem.

(c) We will prove the backward direction by induction on $|X|$. Prove the base case, and state the inductive hypothesis.

(d) Prove the inductive step under the stronger assumption that $|T| < |N(T)|$ for $T \subset X$. (This leaves only the case $|T| = |N(T)|$ remaining.)

(e) Next, suppose that $B \subset X$ satisfies $|B| = |N(B)|$. Let $B' = X \setminus B$, and let $A = N(B)$ and $A' = Y \setminus A$. Draw a diagram for G labeling these sets.

(f) Show that the restrictions of G to $B \cup A$ and $B' \cup A'$ (that is, the graphs obtained from each set by deleting the remaining vertices) satisfy the indutive hypothesis.

(g) Conclude that Hall's Marriage Theorem holds.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the chromatic polynomial of $K_{3,3}$.

(H2) Fix a graph G with n vertices and m edges.

(a) Prove that the leading coefficient of $\chi_G(k)$ (that is, the coefficient of k^n) is 1.

(b) Prove that the coefficient of k^{n-1} in the chromatic polynomial $\chi_G(k)$ equals $-m$.

Hint: each part of this problem can be proven in (at least) 2 distinct ways. One way uses a direct proof with Problem (D1)(b), and another uses induction with Problem (D1)(e).

(H3) Fix a connected graph G with n vertices. Prove that G is a tree if and only if

$$\chi_G(k) = k(k-1)^{n-1}.$$

Hint: use Problem (H2) for the backwards direction.

(H4) Which of the following can be the degrees of the vertices of a bipartite graph?

(a) $3, 3, 3, 3, 3, 3$ (6 vertices total)

(b) $3, 3, 3, 3, 3, 3, 3, 3$ (8 vertices total)

(c) $3, 3, 3, 3, 3, 5, 6, 6, 6$ (9 vertices total)

(H5) Suppose G is a bipartite graph in which every vertex has degree k. Prove that G has a perfect matching.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $n \geq 1$ and $k < n/2$. Let $G = (X, Y)$ denote a bipartite graph so that X consists of the k-element subsets of $[n]$, Y consists of the $(k+1)$-element subsets of $[n]$, and $(x, y) \in E(G)$ for $x \in X$ and $y \in Y$ precisely when $x \subset y$. Find a perfect matching for X into Y.

(C2) Solve all parts of Problem (H2) using a **different** method than you already did.