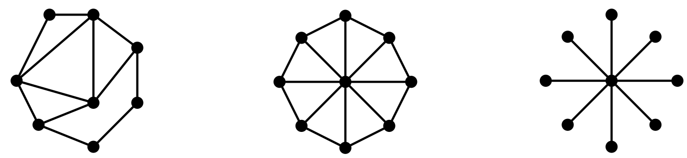
Fall 2019, Math 579: Problem Set 8
Due: Tuesday, October 29th, 2019
Planar Graphs

Discussion problems. The problems below should be completed in class.

(D1) *Counting faces of planar graphs.* For a planar graph G, let V, E, and F denote the number of vertices, edges, and faces of G, respectively.

(a) Compute the quantity $V - E + F$ for each of the following graphs.



(b) Have each group member draw their favorite connected planar graph with at least 8 vertices and 15 edges, and compute $V - E + F$ for their graph.

(c) Notice this came out the same for each graph. This is known as *Euler's theorem* for planar, connected graphs. We will prove this by induction on E.

(i) Base case: prove Euler's theorem when $E = V - 1$. Why is this the base case?

(ii) **Carefully and precisely**, write the inductive hypothesis.

(iii) What can happen when an edge $e \in E(G)$ is removed?

(iv) Finish your proof that Euler's theorem holds for any planar graph G.

(d) What equation must V and E satisfy if G is self-dual (that is, if G is isomorphic to one of its dual)? Is it true that G is self-dual if and only if this equation holds?

(e) Use Euler's Theorem to give a non-pictorial proof that K_5 is not planar. Hint: how many sides would each face need to have?

(f) Use Euler's Theorem to give a non-pictorial proof that $K_{3,3}$ is not planar.

(g) Fix a simple (not necessarily planar!) graph G with V vertices and E edges.

(a) Prove that if G is planar, then $3F \leq 2E$.

(b) Prove that if G is planar, then $E \leq 3V - 6$.

(c) Is it true that any connected graph satisfying $E \leq 3V - 6$ is planar?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that the wheel graph W_n is self-dual.

(H2) Prove that if any 2 edges are removed from the graph K_6, the result is not planar. Is the same true if we remove 3 edges?

(H3) Suppose for a given planar graph G, each face of G (including the "outside" one!) has either 3 or 5 boundary edges. Prove that the number of faces of G is even.