Fall 2019, Math 579: Problem Set 10
Due: Thursday, November 21st, 2019
Operations on Power Series

Discussion problems. The problems below should be worked on in class.

(D1) *Power series of polynomial functions.* Recall that in lecture, we saw

$$1 + 2z + 3z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2},$$

and that the "formal derivative" of $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is

$$A'(z) = \frac{d}{dz}A(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n.$$

(a) Manipulate the first expression above to write $\sum_{n=0}^{\infty} nz^n$ as a rational expression in z.

(b) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^2 z^n$ as a rational expression in z.

(c) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^3 z^n$ as a rational expression in z.

(D2) *Consistency.* The goal of this problem is to do some "sanity checks" on formal power series derivative rules. Recall the power series

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}z^n.$$

(a) Verify the term-by-term derivative of the series for e^{2z} equals the series for $2e^{2z}$.

(b) Find the derivative of $z^2 e^z$ using (i) term-by-term differentation, and (ii) using standard derivative rules, and verify they are equal.

(c) Verify that

$$\frac{e^z - 1}{1 - z} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k!}\right)z^n$$

(d) Verify that the term-by-term derivative of the series in part (c) matches the one obtained from quotient rule.

(D3) *Power series derivatives.* Fix formal power series $A(z)$ and $B(z)$. The goal of this problem is to prove some of the "derivative rules" from Calculus.

(a) Prove that $\frac{d}{dz}[A(z)B(z)] = A'(z)B(z) + A(z)B'(z)$.

(b) Prove that if $B(z)$ has constant term **zero**, then $\frac{d}{dz}[A(B(z))] = A'(B(z))B'(z)$.

(c) Prove that if $B(z)$ has a **nonzero** constant term, then

$$\frac{d}{dz}\left[\frac{A(z)}{B(z)}\right] = \frac{A'(z)B(z) - A(z)B'(z)}{(B(z))^2}.$$

Hint: using parts (a) and (b) can save a LOT of algebra.

(d) Where in parts (b) and (c) did you use the assumption(s) on $B(z)$?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let $A(z) = \sum_{n=0}^{\infty}(n^2 + 3n - 5)z^n$. Find a rational expression for $A(z)$ in terms of z. Your answer should **not** contain any "$\cdots$" or sigma-sums.

Hint: you already did most of the work in discussion problem (D1)!

(H2) Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 1$, and $a_n = na_{n-1} + n(n-1)a_{n-2}$ for $n \geq 2$. Hint: use exponential generating functions.

(H3) Let

$$A(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$

$$B(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots$$

(a) Show that $A'(z) = B(z)$ and $B'(z) = -A(z)$.

(b) Using part (a), show that $2A(z)A'(z) + 2B(z)B'(z) = 0$.

(c) Use part (b) to argue that $(A(z))^2 + (B(z))^2 = 1$. Thinking back to Calculus 2, what familiar identity does this equality encode?

(H4) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$. Prove that $A(z)$ has a multiplicative inverse if and only if $a_0 \neq 0$. Hint: if $A(z)B(z) = 1$ for some $B(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, show each b_n is determined by $a_0, a_1, \ldots, a_n$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$. Prove that we can express the generating function

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n = \frac{Q(z)}{(1-z)^{d+1}}$$

for some polynomial $Q(z)$ of degree at most d if and only if $f(n) = c_d n^d + \cdots + c_1 n + c_0$ is a polynomial of degree exactly d.