Fall 2019, Math 579: Problem Set 11
Due: Thursday, December 5th, 2019
Combinatorics of Ordinary Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Multiplying ordinary generating functions.* Recall that for any two generating functions $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$, we have

$$A(z)B(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n,$$

where the coefficients can be interpreted as the number of ways to split $[n]$ into two (possibly empty) subintervals $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, place an "a-structure" on the first subinterval, and a "b-structure" on the second subinterval.

(a) Let $p(n)$ denote the number of partitions of the integer n (that is, the number of ways to write n as a sum of (unordered) positive integers). Recall from Tuesday that

$$P(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{i=1}^{\infty} \frac{1}{1 - z^i}.$$

Find (by manually counting) the first 3 nonzero coefficients of the generating functions $O(z) = \sum_{n=0}^{\infty} p_{\text{odd}}(n) z^n$ and $E(z) = \sum_{n=0}^{\infty} p_{\text{even}}(n) z^n$, where $p_{\text{odd}}(n)$ and $p_{\text{even}}(n)$ count the number of partitions of n into all odd parts and all even parts, respectively.

(b) Find expressions for $O(z)$ and $E(z)$ in terms of z.

(c) Use the product formula (not algebra!) to give a combinatorial proof $P(z) = O(z)E(z)$.

(d) Use the product formula to find an **expression for the generating function** of the number c_n of ways to take lineup of n soldiers, split them into 2 squadrons, and choose a leader from within the first squadron. Then, find a formula for c_n.

(e) What if the second squadron must have an even number of soldiers in it?

(D2) *Composing ordinary generating functions.*

Theorem. *Suppose a_n (resp. b_n) denotes the number of ways to put an a-stucture (resp. b-structure) on the set $[n] = \{1, 2, \ldots, n\}$, and suppose $b_0 = 0$. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$. The coefficient of z^n in the composition*

$$A(B(z)) = \sum_{n=0}^{\infty} a_n (B(z))^n$$

equals the number of ways to split the set $[n]$ into some number of nonempty subintervals, place a b-structure on each subinterval, and place an a-structure on the set of subintervals.

(a) Find (algebraically) the coefficients of $A(A(z))$, where $A(z) = \sum_{n=1}^{\infty} z^n$. Looking back at Problem (P2), interpret the coefficients of $A(A(z))$ combinatorially.

(b) Let $B(z) = 1/(1-z)$. Interpret the coefficients of $A(B(z))$ combinatorially. (Note that this is not valid since $b_0 \neq 0$. Reconcile this with your answer.)

(c) Let $c_n = n$ denote the number of ways to pick a person from a lineup of n people, and let $C(z) = \sum_{n=1}^{\infty} c_n z^n$ denote its generating function. With $A(z) = \sum_{n=1}^{\infty} z^n$ as above, describe what the coefficients of $C(A(z))$ represent, and find a formula using algebra.

(d) Interpret the coefficients of $A(C(z))$, then find a formula.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let m_n denote the number of ways to express $n \geq 0$ as a sum the integers 6, 9 and 20. For instance, $m_{18} = 2$ since $18 = 6 + 6 + 6 = 9 + 9$, and $m_{26} = 1$ since $26 = 6 + 20$ is the only way to write 26 as such a sum. Find a rational expression for the ordinary generating function of m_n. Do **not** attempt to find a formula for m_n!

(H2) For each of the following, find the an expression for the (ordinary) generating function $D(z) = \sum_{n=0}^{\infty} d_n z^n$ using the combinatorial interpretation of multiplication and composition of power series. Note: your final answer to each part should be a rational expression for $D(z)$ in terms of z, and should not have any infinite sums or products. In particular, you are **not** required to find a formula for d_n!

- (a) Suppose our course meets for n days. Let d_n denote the number of ways to split the available days into 3 units (counting methods, graph theory, and generating functions),
  - (i) select some (possibly empty) collection of days from the first unit for a pop quiz,
  - (ii) select an odd number of days from the second unit to hold discussions, and
  - (iii) select a single day from the third unit in which to give an exam.
- (b) What if in part (a), each unit should be at least 2 days long?
- (c) What if in part (a), exactly half of the days in the first unit must have a pop quiz, and they must alternate days (quiz, no quiz, quiz, no quiz, etc)?

(H3) Choose **one** of the parts of the previous problem, and use the generating function expression you obtained to find a formula for d_n. You may use a computer (e.g., `WolframAlpha`) to perform the partial fractions step for you.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) For each integer $t \geq 0$, consider the generating function

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} z^n.$$

Prove that $B_n(t)$ is a polynomial in t of degree n.