Fall 2019, Math 579
Midterm Exam 3 Review

The problems below are intended to help you review for the midterm exam, and may *not* be turned in for credit.

(ER1) In defining the composition $A(B(z))$ of formal power series, we require that $B(z)$ have constant term 0. Why is this?

(ER2) Derive a simple expression for the ordinary generating function of the sequence $a_n = n^2$. Do the same for its exponential generating function.

(ER3) Use ordinary generating functions to find a closed form for the recurrence relation given by $a_0 = 1$ and $a_n = 2a_{n-1} + n^2$.

(ER4) Use exponential generating functions to find a closed form for the recurrence relation given by $a_0 = 1$ and $a_n = na_{n-1} + n!$ for $n \geq 1$.

(ER5) Fix power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$, and let $\frac{d}{dz}$ denote term-by-term differentation. Verify

$$\tfrac{d}{dz}\left(A(z)B(z)\right) = \left(\tfrac{d}{dz}A(z)\right)B(z) + A(z)\left(\tfrac{d}{dz}B(z)\right),$$

the product rule for formal power series.

(ER6) Recall that $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, and consider the formal power series

$$\ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1}.$$

(a) Justify the above definition by differentiating both sides (using calculus).
(b) Prove $e^{-z} = 1/e^z$.
(c) Prove that $\ln(e^z) = z$.
(d) Pick your favorite algebraic property involving $\ln(1+z)$ and/or e^z, and prove that it holds in formal power series-land. Alternatively, look up the power series expansions for $\sin(z)$ and $\cos(z)$ in your favorite Calculus textbook, and prove (using formal power series) that $\sin(2z) = 2\sin(z)\cos(z)$.

(ER7) Recall that $S(n,k)$ denotes the number of **set** partitions of $[n]$ into k nonempty blocks. For example, $S(3,2) = 3$ counts the partitions $\{\{12\},\{3\}\}$, $\{\{13\},\{2\}\}$, and $\{\{23\},\{1\}\}$ of $[3]$ into exactly 2 blocks.

For $k \geq 1$, find an expression for the exponential generating function

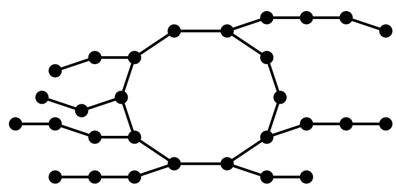
$$S_k(z) = \sum_{n=0}^{\infty} S(n,k)\frac{z^n}{n!}$$

in terms of familiar exponential generating functions (e^z, $\ln(z)$, etc.). Use this to find a closed form when $k = 1$, $k = 2$ and $k = 3$. Note: there is no known closed form for general k, so do not attempt to solve for the coefficients in general!

(ER8) A permutation of $[n] = \{1, \ldots, n\}$ is called *indecomposable* if it cannot be split into a permutation on $\{1, \ldots, k\}$ and a permutation on $\{k+1, \ldots, n\}$ for $1 \leq k \leq n-1$. For example, 54321 is indecomposable, but 23154 can be decomposed as 231 and then 54.

Let c_n denote the number of indecomposable permutations on $[n]$, and let $c_0 = 0$. Find an equation relating the ordinary generating function $C(z)$ for c_n and the ordinary generating function for the number of permutations of $[n]$, that is, $P(z) = \sum_{n=0}^{\infty}(n!)z^n$.

(ER9) A *combinatorial octopus* is an undirected graph of the following form.



Let o_n denote the number of combinatorial octopii on n vertices. Additionally, let ℓ_n denote the number of ways to linearly order a set with n elements, and let c_n denote the number of ways to arrange n elements in a circle. Express the exponential generating function for o_n in terms of the exponential generating functions of ℓ_n and c_n. Use this to find o_n.