Fall 2019, Math 620: Week 2 Problem Set
Due: Thursday, September 12th, 2019
Permutation Groups

Discussion problems. The problems below should be completed in class.

(D1) *Working with permutations.* Consider the following permutations.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 6 & 3 & 8 & 2 & 5 & 7 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 8 & 6 & 1 & 4 & 7 & 3 \end{pmatrix}$$

(a) Write each of the above permutations as a product of disjoint cycles.

(b) Find the order of each element above. Hint: do the disjoint cycles help?

(c) Write each of the above permutations as a product of 2-cycles in two different ways.

(d) Determine which of the above permutations are even, and which are odd.

(e) Is it possible to write σ or τ as a product of disjoint 2-cycles?

(f) Write σ^{-1}, τ^{-1}, and α^{-1} as a products of disjoint cycles, and as products of 2-cycles. Hint: you have already written σ, τ, and α in these forms!

(g) Determine whether σ^{91} is even or odd.

(h) Determine the orders of σ, τ, and α.

(D2) *Group elements as permutations.* The goal of this problem is to develop a proof of the following theorem, as well as intuition for why it holds.

Theorem. *Every finite group G with $|G| = n$ is isomorphic to a subgroup of S_n.*

(a) The goal of the first few parts is to identify a subgroup of S_6 isomorphic to D_3. Choose a labeling of the elements of D_3 as $a_1, a_2, \ldots, a_6$ (you may do this in any way you wish).

(b) Let $r \in D_3$ denote clockwise rotation by $120°$. Define a permutation $\sigma \in S_6$ given by $\sigma(i) = k$ where $a_k = a_i r$. Verify that σ is indeed a permutation by writing it in permutation notation.

(c) Repeat the previous part for each $a \in D_3$ (that is, define a permutation $\sigma_a \in S_6$ given by $\sigma(i) = k$ where $a_k = a_i a$). Write down all 6 resulting permutations (you may want to "divide and conquer" amongst your groupmates to save time!).

(d) Verify in three examples that for any $a, b \in D_3$, the permutation corresponding to ab equals the product of the permutations corresponding to a and b.

(e) Using the ideas above, find a subgroup of S_4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(f) Write a formal proof of the theorem at the start of this problem.

(D3) *Prove or disprove.* Determine whether each of the following statements is true or false.

(a) For every $n, m \geq 2$ with $m \mid n!$, there exists an element of S_n of order m.

(b) There exists a subgroup of S_6 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

(c) There exists a subgroup of S_6 isomorphic to $\mathbb{Z}_{12}$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Identify an element of S_9 of order 20.

(H2) Let $B_n \subset S_n$ denote the set of odd permutations in S_n. Prove that $|B_n| = |A_n| = n!/2$. Hint: can you find a bijection $f : A_n \to B_n$?

(H3) Prove S_n is isomorphic to a subgroup of A_{n+2}.

(H4) Prove that a permutation $\sigma \in S_n$ is a product of disjoint 2-cycles if and only if $\sigma = \sigma^{-1}$.

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) For each $n \geq 3$, every permutation in S_n can be written as a product of 3-cycles.

(b) For each $n \geq 2$, every permutation in S_n is a product of at most $n - 1$ transpositions.

(c) For each $n \geq 3$, every permutation in S_n is a product of adjacent transpositions.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a generating set for S_n consisting of only 2 generators.