Fall 2019, Math 620: Week 3 Problem Set
Due: Thursday, September 19th, 2019
Cosets and Quotient Groups

Discussion problems. The problems below should be completed in class.

(D1) *Free groups and relations.* Define the *free group on k generators*, denoted $F = \langle a_1, \ldots, a_k \rangle$, whose elements have the form $s_1^{e_1} \cdots s_m^{e_m}$ where each $s_i \in \{a_1, \ldots, a_k\}$ and each $e_i = \pm 1$ (called *words* in the *letters* $a_1, \ldots, a_k$). For example,

$$a \qquad ab \qquad ba \qquad aabbb \qquad aab \qquad abb^{-1}ab$$

are all elements of $F = \langle a, b \rangle$, and only the last 2 are equal. Words are sometimes written more succinctly using higher exponents, e.g., $a^2b^3a^{-4} = aabbba^{-1}a^{-1}a^{-1}a^{-1}$. The *normal subgroup generated by* $w_1, \ldots, w_r \in F$, denoted $H = \langle w_1, \ldots, w_r \rangle$ (note the potential ambiguity with the $\langle \cdot \rangle$ notation!), is the smallest normal subgroup of F containing $w_1, \ldots, w_r$. The elements of H are products of conjugates of the w_i's and their inverses, e.g.,

$$\left(bw_1b^{-1}\right) \left((ab)w_1^{-1}(ab)^{-1}\right) \left((b^2a^{-1})w_3^{-1}(b^2a^{-1})^{-1}\right) \left(aw_1a^{-1}\right) \in H.$$

Intuitively, the elements of F/H are elements of F where we force each $w_i = e$ (we call each w_i a *relation* in F/H).

(a) Let $F_1 = \langle a \rangle$ denote the free group on 1 generator. Prove $F_1 \cong \mathbb{Z}$.

(b) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators, and let $H = \langle aba^{-1}b^{-1} \rangle$. Explain intuitively why $ab = ba$ in $G = F_2/H$, then prove it using the definitions above.

(c) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators, and let $H = \langle a^2, b^2, aba^{-1}b^{-1} \rangle$. How many elements does $G = F_2/H$ have?

(d) In the previous 2 examples, what more familiar groups are isomorphic to $G = F_2/H$?

(e) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators. For each of the following normal subgroups H and H', prove $H = H'$ by representing each generator of H as a product of conjugates of generators of H', and visa versa.

(i) $H = \langle a^3b^2 \rangle$, $H' = \langle ba^3b \rangle$.
(ii) $H = \langle aba^{-1}b^{-1}, a^2b^3ab^4 \rangle$, $H' = \langle aba^{-1}b^{-1}, a^3b^7 \rangle$.
(iii) $H = \langle a^4ba^{-3}b^{-2}, a^3b^2a^{-2}b^{-3}, a^2b^3a^{-1}b^{-4} \rangle$,
$H' = \langle a^4ba^{-3}b^{-2}, a^3b^2a^{-2}b^{-3}, a^4ba^{-1}b^{-4} \rangle$.

(f) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators, and let $H = \langle a^4, b^2, abab \rangle$. Prove $G = F_2/H$ has 8 elements, and find a representative of each.

(g) Prove the group from the previous part is isomorphic to D_4.

(h) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators, let $H = \langle a^4, b^4, a^2b^{-2}, abab^{-1} \rangle$, and let $G = F_2/H$. Find $|G|$, and determine whether G is Abelian.

(i) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators, and choose $\sigma, \tau \in S_3$ that generate S_3. Locate a normal subgroup H so that $F_2/H \cong S_3$ with $a \mapsto \sigma$ and $b \mapsto \tau$.

(j) Let $F_2 = \langle a, b \rangle$ and $F_3 = \langle x, y, z \rangle$ denote the free groups on 2 and 3 words, respectively. Locate a (not necessarily normal) subgroup of F_2 that is isomorphic to F_3.

(k) Prove or disprove: $(\mathbb{Q}, +)$ is a quotient of the free group on 2 generators.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) What more familiar group is isomorphic to $GL_2(\mathbb{Z}_2)$?

(H2) Suppose G is a group and $H \subset G$ is a subgroup. Prove $b \in aH$ if and only if $bH = aH$.

(H3) Prove A_n is a normal subgroup of S_n.

(H4) Fix a group G. Define the *center* of G as the set

$$C = \{c \in G : ca = ac \text{ for all } a \in G\}$$

of elements that commute with every element of G.

(a) Prove C is a normal subgroup of G.

(b) Prove or disprove: the center of G/C is trivial.

(H5) Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators, and suppose $H \subset F_2$ is a normal subgroup. Prove F_2/H is Abelian if and only if $aba^{-1}b^{-1} \in H$.

(H6) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Up to isomorphism, the only non-Abelian group of order 8 is D_4.

(b) Every finite group of prime order (i.e., prime cardonality) is cyclic.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose p is prime. Find all groups G (up to isomorphism) with $|G| = p^2$.