Fall 2019, Math 620: Week 4 Problem Set
Due: Thursday, September 26th, 2019
Isomorphism Theorems

Discussion problems. The problems below should be worked on in class.

(D1) *The third isomorphism theorem.* The goal of this problem is to prove the following theorem. Tip: due to the large number of quotients in this problem, avoid using "bars" (e.g., $\overline{a}$).

Theorem. *If G is a group and $K, H \lhd G$ with $K \subset H$, then $(G/K)/(H/K) \cong G/H$.*

(a) Let $G = \mathbb{Z}_{24}$, $H = \{[3k]_{24} : k \in \mathbb{Z}\}$, and $K = \{[12k]_{24} : k \in \mathbb{Z}\}$. Verify the above theorem holds in this case.

(b) First, we must identify a subgroup $H' \subset G/K$ that corresponds to H/K in the "large" quotient above. Identify a "natural" candidate for H', and prove it is normal subgroup.

(c) Propose a map $\varphi : (G/K)/H' \to G/H$ to serve as a starting place for the isomorphism. Why is your map a natural choice?

(d) Prove your map φ is well defined.

(e) Prove your map φ is a bijection.

(f) Complete the proof by verifying φ is a homomorphism.

(g) Can the above proof be shortened using the first isomorphism theorem? In particular, can we define a homomorphism $\phi : G/K \to G/H$ whose kernel equals H'?

(D2) *Subgroup correspondence.* The goal of this problem is to discover the "correspondence theorem" for subgroups of quotient groups.

(a) Find all subgroups of $\mathbb{Z}_{72}$, organized by containment. Do the same for $\mathbb{Z}_{24}$.

(b) The quotient of $\mathbb{Z}_{72}$ by which subgroup is isomorphic to $\mathbb{Z}_{24}$? Do you notice any relationship between the arrangement of subgroups of $\mathbb{Z}_{24}$ and some subgroups of $\mathbb{Z}_{72}$?

(c) Fix a group G and a normal subgroup $H \lhd G$. Formulate a conjecture specifying a bijection between the subgroups of G/H and certain subgroups of G. Use the map $G \to G/H$ to specify the bijection.

(d) Prove your conjecture.

(e) Find all subgroups of the group D_4. Use your list to determine how many subgroups the quotient group $D_4/\{e, r^2\}$ has.

(D3) *Prove or disprove.* Determine whether each of the following statements is true or false.

(a) If G is a group and $H, K \lhd G$ with $H \cap K = \{e\}$, then $G \cong G/H \times G/K$.

(b) The only element of $\mathbb{R}/\mathbb{Z}$ with finite order is the identity.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose G is a group. Given $a \in G$, define $f_a : G \to G$ by $f_a(x) = aba^{-1}$.

(a) Prove f_a is an automorphism (these are known as *inner automorphisms*).

(b) Let $G' = \{f_a : a \in G\} \subset \mathrm{Aut}(G)$. Prove G' is a normal subgroup of $\mathrm{Aut}(G)$.

(c) Let $\varphi : G \to G'$ denote the map $a \mapsto f_a$. Characterize the elements of G in $\ker(\varphi)$.

(d) Characterize which groups G have a unique inner automorphism.

(H2) The following questions pertain to the correspondence theorem from Problem (D2).

(a) If a subgroup of G/H is normal, is its corresponding subgroup of G normal? What about the converse of this statement?

(b) If a subgroup of G/H is cyclic, is its corresponding subgroup of G cyclic? What about the converse of this statement?

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If G is a group and $H, K \lhd G$, then $G/(H \cap K) \cong G/H \times G/K$.

(b) If G is a group and $H, K \lhd G$ with $K \subset H$, then $G/H \times H/K \cong G/K$.

(c) If G is a group and $H, K \lhd G$ with $H \cap K = \{e\}$ and $HK = G$, then $G \cong G/H \times G/K$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Given a finite Abelian group G, find a formula for $|\,\mathrm{Aut}(G)|$.