Fall 2019, Math 620: Week 5 Problem Set
Due: Thursday, October 3rd, 2019
Introduction To Rings

Discussion problems. The problems below should be worked on in class.

(D1) *Checking ring axioms.* Determine which of the following sets R is a ring under the given addition and multiplication. For each ring, determine whether it is (i) commutative, (ii) an integral domain, and (iii) a field.

Hint: first, write all axioms on the board for reference, and decide which axioms are satisfied "for free" when the proposed ring is a subset of a known ring with identical operation(s).

(a) The set R of 2×2 real matrices (under matrix addition/multiplication) given by

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subset GL_2(\mathbb{R}).$$

(b) The set $R = \{r_5 x^5 + \cdots + r_1 x + r_0 : r_i \in \mathbb{R}\} \subset \mathbb{R}[x]$ of polynomials in a variable x with real coefficients and **degree at most 5**, under the usual addition and multiplication.

(c) The set $R = \mathbb{R} \cup \{\infty\}$ of real numbers together with infinity, and addition and multiplication operations $a \oplus b = \min(a, b)$ and $a \odot b = a + b$, respectively.

(d) The set $R = \mathbb{Z}$ with operations $\oplus$ and $\odot$ given by $a \oplus b = a + b$ and $a \odot b = a + b$ (in particular, **both** addition and multiplication in R correspond to integer addition).

(e) The set $R = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of continuous real-valued functions on $\mathbb{R}$, where addition $+$ is the usual addition of functions, and multiplication $\odot$ is given by composition, e.g. $\sin(x) \odot e^x = \sin(e^x)$.

(f) The set $R = \{p(x) \in \mathbb{R}[x] : p(0) \in \mathbb{Z}\}$ of polynomials in a variable x with real coefficients and **integer constant term**, under the usual addition and multiplication. For example, $2x^2 + \frac{1}{2}x + 5 \in R$ and $\frac{6}{5}x \in R$, but $5x + \frac{1}{3} \notin R$.

(D2) *Cartesian products.* The Cartesian product of two rings R_1 and R_2 is the set

$$R_1 \times R_2 = \{(a, b) : a \in R_1, b \in R_2\}$$

with addition $(a, b) + (a', b') = (a + a', b + b')$ and multiplication $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$. Note: the operation in each coordinate happen in their respective rings.

(a) Determine which elements of $\mathbb{Z}_5 \times \mathbb{Z}_4$ are units, and which are zero-divisors.

(b) Suppose $m, n \geq 2$. Determine the units and zero-divisors of $\mathbb{Z}_m \times \mathbb{Z}_n$.

(c) Suppose R_1 and R_2 are rings. Determine which elements of $R_1 \times R_2$ are units, in terms of the units of R_1 and the units of R_2.

(d) Suppose R_1 and R_2 are rings. Determine which elements of $R_1 \times R_2$ are zero-divisors, in terms of the zero-divisors of R_1 and the zero-divisors of R_2.

(e) Suppose R_1 and R_2 are rings. Can there be elements of $R_1 \times R_2$ that are neither units nor zero-divisors?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let $R = \mathbb{Z}$ and define

$$a \oplus b = a + b + 1 \qquad \text{and} \qquad a \odot b = ab + a + b$$

for all $a, b \in R$. Prove that $(R, \oplus, \odot)$ is a commutative ring. Is R a field?

(H2) Consider $(C, +, \odot)$, where $C = \mathbb{R}\times\mathbb{R}$, "$+$" is the standard componentwise addition on $\mathbb{R}\times\mathbb{R}$, and "$\odot$" is given by

$$(a, b) \odot (c, d) = (ac - bd, ad + bc)$$

for all $(a, b), (c, d) \in C$. Prove that C is a field. Hint: this can be done **without** manually verifying axioms by proving straight away that C is isomorphic to a more familiar field.

(H3) Suppose $R = \{0_R, 1_R, a\}$ is a ring with 3 distinct elements. Use the ring axioms to fill in the addition table and multiplication table of R. Give a justification for each entry.

(H4) Suppose $(R, +, \cdot)$ is a ring. Prove each of the following. Identify each ring axiom you use, and try to only use one axiom in each step.

(a) If $a, b, c \in R$ with $ab = 1$ and $ca = 1$, then $b = c$.

(b) If R has unity and $1 = 0$, then $R = \{0\}$.

(H5) Determine for which $m \geq 2$ the set of non-unit elements of $\mathbb{Z}_m$ is closed under both addition and multiplication. Prove your claim.

(H6) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If R is a commutative ring and $a, b \in R$ are zero divisors, then ab is a zero divisor.
Hint: this one is subtle!

(b) Let S be a set and $P(S)$ denote the set of all subsets of S. Define addition and multiplication operations $\oplus$ and $\odot$ on $P(S)$ by

$$M \oplus N = (M \setminus N) \cup (N \setminus M) \qquad \text{and} \qquad M \odot N = M \cap N$$

for all $M, N \in P(S)$. Determine whether $(P(S), \oplus, \odot)$ is a field.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a ring R with unity 1_R and a subring $S \subset R$ with unity 1_S such that $1_R \neq 1_S$ (that is, the unity of S is a **different element** than the unity of R).