

Fall 2019, Math 620: Week 7 Problem Set
Due: Tuesday, October 15th, 2019
Rings of Fractions and Localization

Discussion problems. The problems below should be completed in class.

- (D1) *Constructing the rationals from the integers.* Define an equivalence relation \sim on $\mathbb{Z} \times \mathbb{Z}_{\neq 0}$ by $(a, b) \sim (a', b')$ when $ab' = a'b$. Let Q denote the set of equivalence classes of \sim . Define operations \oplus and \odot on Q so that

$$[(a, b)] \oplus [(c, d)] = [(ad + bc, bd)] \quad \text{and} \quad [(a, b)] \odot [(c, d)] = [(ac, bd)]$$

for all $[(a, b)], [(c, d)] \in Q$. (Intuitively, $[(a, b)] \in Q$ represents $a/b \in \mathbb{Q}$.)

- (a) Prove addition is well-defined in Q , that is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(a, b)] \oplus [(c, d)] = [(a', b')] \oplus [(c', d')]$. Do the same for multiplication.
 - (b) Prove addition in Q is commutative and associative.
 - (c) Prove multiplication in Q is commutative and associative.
 - (d) Prove distributivity holds in Q .
 - (e) Prove every element of Q has an additive inverse, and that every nonzero element has a multiplicative inverse.
- (D2) *Constructing the integers from the naturals.* Define an equivalence relation \sim on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \sim (a', b')$ when $a + b' = a' + b$. Let Z denote the set of equivalence classes of \sim . Define operations \oplus and \odot on Z so that

$$[(a, b)] \oplus [(c, d)] = [(a + c, b + d)] \quad \text{and} \quad [(a, b)] \odot [(c, d)] = [(ac + bd, ad + bc)]$$

for all $[(a, b)], [(c, d)] \in Z$. (Intuitively, $[(a, b)] \in Z$ represents $a - b \in \mathbb{Z}$.)

- (a) Prove addition is well-defined in Z , that is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(a, b)] \oplus [(c, d)] = [(a', b')] \oplus [(c', d')]$. Do the same for multiplication.
 - (b) Prove addition in Z is commutative and associative.
 - (c) Prove multiplication in Z is commutative and associative.
 - (d) Prove distributivity holds in Z .
 - (e) Prove every element of Z has an additive inverse.
- (D3) *Constructing the natural numbers.* Let N be a set with the following properties:

- there is a distinguished element $0 \in N$;
- every $a \in N$ has a *successor* $a^+ \in N$; and
- 0 is not the successor of any element.

Define the operations \oplus and \odot on N so that

$$a \oplus 0 = a \quad a \oplus b^+ = (a \oplus b)^+ \quad a \odot 0 = 0 \quad \text{and} \quad a \odot b^+ = (a \odot b) \oplus a$$

for all $a, b \in N$. Every proof in this part will use “induction”: if a given statement holds for $0 \in N$ and it holds for a^+ whenever it holds for a , then it holds for every element of N .

- (a) For convenience, let $1 = 0^+$, $2 = 1^+$, and $3 = 2^+$. Find $a \oplus b$ for every $a, b \in \{0, 1, 2, 3\}$.
- (b) Prove that $a^+ \oplus b = (a \oplus b)^+$ for every $a, b \in N$.
- (c) Prove that addition is commutative and associative in N .
- (d) Prove that $a^+ \odot b = (a \odot b) \oplus b$ for every $a, b \in N$.
- (e) Prove that multiplication is commutative and associative in N .
- (f) Prove that distributivity holds in N .

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) In each of the following, you may use “free of charge” (i.e., without proof) any prior parts of the same discussion problem, but **not** any from previous discussion problems.

- (a) Write up a proof for any one part of Problem (D1) you choose.
- (b) Write up a proof for any one part of Problem (D2) you choose (preferably involving different adjective(s) than your choice from (D1)).
- (c) Write up a proof for any one part of Problem (D3)\{(a)\} you choose (preferably involving different adjective(s) than your choices from (D1) and (D2)).

(H2) Fix a ring R and a prime ideal $P \subset R$, and let $D = R \setminus P$.

- (a) Prove that D is a multiplicative set.
- (b) Prove that $D^{-1}R$ has a unique maximal ideal (the ring $R_P = D^{-1}R$ is called the *localization* of R at P and plays an important role in algebraic geometry).