Fall 2019, Math 620: Week 12 Problem Set
Due: Tuesday, December 10th, 2019
Categories and Universal Properties

Discussion problems. The problems below should be completed in class.

(D1) *Kernels.* Recall that the *kernel* of a morphism $f : A \to B$ in a category $\mathcal{C}$ is an object K together with a morphism $g : K \to A$ such that (i) the composition $f \circ g = 0$, and (ii) for any object K' and morphism $g' : K' \to A$ with $f \circ g' = 0$, there exists a unique morphism $h : K' \to K$ such that $g \circ h = g'$.

(a) **Without looking at your notes**, draw the commutative diagram for kernels.

(b) Let $\varphi : \mathbb{Z} \to \mathbb{Z}_{12}$ denote the group homomorphism given by $\varphi(a) = [4a]_{12}$. Prove that $K = \ker \varphi$ with the inclusion map $g : K \hookrightarrow \mathbb{Z}$ satsifies the universal property above.

(c) Prove that every morphism in the category Ab has a (categorical) kernel that coincides with what we have been calling the "kernel" all semester.

(d) Prove that in any category $\mathcal{C}$ in which every morphism has a kernel (as defined above), the morphism g must be a monomorphism. Be careful: morphisms might not be "functions" in general!

(D2) *Products.* Recall the *product* $A \prod B$ of two objects A and B in a category $\mathcal{C}$, defined below.

$$\begin{tikzcd} & C \arrow[dl, "f'"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "g'"] & \\ A & A \times B \arrow[l, "f"] \arrow[r, "g"'] & B \end{tikzcd}$$

(a) **Without looking at your notes**, write the full definition of $A \times B$ **in words**.

(b) Fix an index set $\mathcal{I}$ and a collection of objects A_i for $i \in \mathcal{I}$ in a category $\mathcal{C}$. Give a definition of the *product* $\prod_{i \in \mathcal{I}} A_i$. Use both words and a commutative diagram.

(c) In the category Ab, prove that (categorical) products are simply Cartesian products.

(D3) *Duals.* Given a universal object (e.g., kernels, products), the *dual* is obtained by reversing the directions of all of the arrows in the accompanying commutative diagram.

(a) Write the definition of *coproduct* (the categorical dual of the product) of two objects A and B in a category $\mathcal{C}$. Denote this object by $A \amalg B$.

(b) In the category Ab, determine what familiar group $\mathbb{Z} \amalg \mathbb{Z}_6$ is.

(c) Demonstrate that in the category Ab, finite coproducts are simply finite products.

(d) Determine whether $\coprod_{i=1}^{\infty} \mathbb{Z}$ is isomorphic to $\prod_{i=1}^{\infty} \mathbb{Z}$ in Ab.

(e) Conjecture which familiar objects are coproducts in Ab.

(f) Write the definition of *cokernel* (the categorical dual of the kernel) of two objects A and B in a category $\mathcal{C}$.

(g) Find the cokernel of the $\mathbb{Z}$-module homomorphism $f : \mathbb{Z} \to \mathbb{Z}_{12}$ given by $f(a) = [4a]$.

(h) Conjecture, in general, what the cokernel of an R-module homomorphism is.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Given a category $\mathcal{C}$ and a morphism $f : A \to B$, the *image* of f (if it exists) is an object I together with a monomorphism $m : I \to B$ such that

- there exists a map $e : A \to I$ such that $m \circ e = f$; and
- the following universal property is satisfied: for any object I', morphism $e' : A \to I'$, and monomorphism $m' : I' \to B$ satisfying the above requirements, there exists a unique morphism $v : I \to I'$ such that the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "e"] \arrow[drr, bend left, "e'"] & & \\
& I \arrow[dl, hook, "m"] \arrow[r, dashed, "\exists! v"] & I' \arrow[dll, hook, bend left, "m'"] \\
B & &
\end{tikzcd}
$$

Prove that for any ring R in Slt, the category R-Mod has images.

(H2) Determine whether each of the following statements is true or false. Prove your assertions.

(a) The category Slt has both (arbitrary) products and (arbitrary) coproducts.

(b) The category Slt has kernels.