Fall 2019, Math 620
Midterm Exam Review

The problems below are intended to help you review for the midterm exam, and may *not* be turned in for credit.

(ER1) Prove that the group $G = (\mathbb{Z}_{11} \setminus \{0\}, \cdot\,)$ is cyclic.

(ER2) Find all subgroups of A_4, organized by containment.

(ER3) Determine whether $\{e, r^{24}, f, r^{24}f\}$ is a normal subgroup of D_{48}.

(ER4) Consider the subgroups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R},\ ad \neq 0 \right\} \qquad \text{and} \qquad G' = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R},\ ad \neq 0 \right\}$$

of $GL_2(\mathbb{R})$. Locate a normal subgroup $H \subset G$ so that $G/H \cong G'$.

(ER5) Let $R = \mathbb{R}_{>0}$ denote the set of positive real numbers, and define $a \oplus b = ab$ and $a \odot b = a^{\ln b}$.

(a) Locate the identity elements 0_R and 1_R in $(R, \oplus, \odot)$.
(b) Prove that multiplication in $(R, \oplus, \odot)$ is associative.
(c) Prove that every nonzero element of $(R, \oplus, \odot)$ has a multiplicative inverse.

(ER6) Suppose R is a commutative ring. An element $r \in R$ is *nilpotent* if $r^k = 0$ for some $k \geq 1$.

(a) Prove that the set N of nilpotent elements of R is an ideal (called the *nilradical*).
(b) Prove or disprove: if N is the nilradical of R, then R/N has no zero-divisors.

(ER7) Let $R = \mathbb{Q}[x]$, and consider the ideals $I = \langle x^2 \rangle$ and $J = \langle x^2 - 1 \rangle$. Determine whether R/I and R/J are isomorphic rings.

(ER8) Prove or disprove: if a commutative ring R has finitely many ideals, then R is finite.

(ER9) Suppose R is a commutative ring with unity.

(a) Prove that the set D of units of R is a multiplicative set.
(b) Prove that $R \cong D^{-1}R$.