Fall 2020, Math 579: Problem Set 1
Due: Thursday, September 3rd, 2020
Induction

Discussion problems. The problems below should be worked on in class.

(D1) *Strong induction.* Fix a statement $P(n)$ dependent on n, and suppose that:

- $P(1)$ holds (the *base case*); and
- if $P(k)$ holds for all $1 \leq k \leq n$, then $P(n+1)$ holds (the *inductive step*).

We can conclude that $P(n)$ holds for all n. This technique is called *strong induction* on n. (How does this differ from usual (*weak*) induction?)

(a) Locate and correct the error in the following proof that for any $n \in \mathbb{Z}_{\geq 0}$, there exist $q, r \in \mathbb{Z}_{\geq 0}$ with $0 \leq r \leq 5$ such that $n = 6q + r$.

Proof. Let $P(n)$ denote the following statement.

"There exist $q, r \in \mathbb{Z}_{\geq 0}$ with $0 \leq r \leq 5$ such that $n = 6q + r$."

Base cases: suppose $n = 0, 1, 2, 3, 4$, or 5. Choosing $q = 0$ and $r = n$, we see $6q+r = n$. Inductive step: suppose $n \geq 6$ and that $P(n-6)$ holds (the *inductive hypothesis*). This implies $n - 6 = 6q' + r'$ for some $q', r' \in \mathbb{Z}$ with $0 \leq r' \leq 5$. Rearranging yields $n = 6(q'+1) + r'$, and choosing $q = q' + 1$ and $r = r' + 1$ completes the proof. □

(b) Prove the following results using induction. For each, indicate whether your proof uses strong induction or weak induction.

(i) Prove that for every $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(ii) Define $a_0 = 0$ and

$$a_n = a_0 + \cdots + a_{n-1} + 1$$

for $n \geq 1$. Find a formula for a_n by experimentation, then prove it.

(iii) Prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for all $n \geq 1$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider the function

$$f(n) = 2^2 + 4^2 + \cdots + (2n)^2 = \sum_{i=1}^{n} (2i)^2.$$

(a) Use induction on n to prove that

$$f(n) = \frac{2n(n+1)(2n+1)}{3}.$$

(b) Use Problem (D1)(b)(i) to find an analogous formula for

$$g(n) = 1^2 + 3^2 + \cdots + (2n-1)^2 = \sum_{i=1}^{n} (2i-1)^2$$

in terms of n (induction is **not** required for this problem).

(H2) Suppose $g_0 = 0$, $g_1 = 1$, and $g_n = g_{n-1} + g_{n-2}$ for $n \geq 2$. Use induction to prove that

$$g_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

holds for all $n \geq 0$. Does your proof use strong induction?

(H3) Let $a_0 = 0$ and $a_n = a_{n-1} + 2n - 1$ for $n \geq 1$. Conjecture a formula for a_n in terms of n. Use induction to prove your claim.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) A *round robin tournament* is a tournament in which any 2 players compete against each other exactly once. Prove the round robin tournament theorem: if $n \geq 2$ is even, then there exists an n-player tournament with $n-1$ rounds, and $n \geq 1$ is odd, then there is an n-player tournament with n rounds (each player can play at most one game in a single round).