Fall 2020, Math 579: Problem Set 2
Due: Thursday, September 10th, 2020
Elementary Counting Methods

Discussion problems. The problems below should be worked on in class.

(D1) *Analyzing counting arguments.* Each of the following proofs has an error. Locate the error, explain the issue, and correct the error.

(a) Find the number of ways to choose 3 appetizers and 2 desserts from a menu with 5 appetizers and 6 desserts.

Proof. For the appetizers: there are $\binom{5}{3}$ ways to choose 3 out of the 5 appetizers. For dessert: there are 6 choices for the first dessert and 5 choices for the second dessert, yielding $6 \cdot 5$ possibilities. As such, we obtain

$$\binom{5}{3} \cdot 6 \cdot 5$$

ways to place the full order. □

(b) Find the number of ways to order 112345 so that the 1's **are not** adjacent.

Proof. First, we will count the total number of ways to order 112345. There are 6! ways to order 6 distinct symbols, but since '1' occurs twice, we must divide by 2!. Next, we count the orderings where the 1's **are** adjacent. For this, we can treat '11' as a single symbol, yielding 5! orderings. As such, we obtain

$$\frac{6!}{2! \cdot 5!}$$

total ways when the 1's **are not** adjacent. □

(D2) *Counting practice.*

(a) Find the number of ways to arrange the letters in the word MISSISSIPPI.

(b) Find the number of subsets of [7] with at least 3 elements.

(D3) *Poker hands.* Suppose you have a 52 card deck, with 4 suits (labeled spades, clubs, hearts, diamonds) and 13 ranks (labeled 2 through 10, Jack, Queen, King, and Ace). Each card has one rank and one suit, and no two cards are identical.

(a) Recall the total number of possible 5-card hands (from the preliminary problem).

(b) Determine the number of poker hands that have each ranking:

- royal flush (ranks 10-A, all same suit);
- straight flush (sequential ranks, all same suit);
- 4 of a kind (4 cards have equal rank);
- full house (3 cards have equal rank, the other 2 also have equal rank);
- flush (all cards have the same suit);
- straight (cards have sequential ranks);
- 3 of a kind (3 cards have equal rank);
- 2 pair (2 cards have equal rank, 2 others also have equal rank);
- 1 pair (2 cards have equal rank); and
- high card (none of the above).

Each 5-card hand should fall under exactly one name (e.g., a 2 pair is *not* a pair).

(c) Verify that the sum of all of your answers from part (b) yields the same number as in part (a), and that hands with higher rankings occur less frequently.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Determine the number of ways to place n non-attacking rooks on an $n \times n$ chess board (a rook can move in a straight line up, down, left, and right).

(H2) Prove that exactly half of the subsets of $[n]$ have an even number of elements.

(H3) In how many ways can we select two subsets $C, D \subset [n]$ such that $C \cap D = \emptyset$?

(H4) Suppose $b_1 + \cdots + b_m \leq n$. Prove that $b_1! \cdots b_m! \leq n!$.

(H5) Let P denote a convex n-sided polygon in which no 3 diagonals intersect in a single point. How many intersection points do the diagonals of P have?

(H6) How many $n \times n$ square matrices are there whose entries are 0's and 1's and in which every row and column has an even sum?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) How many non-attacking queens can be placed on an $n \times n$ chess board (queens can move in a straight line up, down, left, right, and diagonal)?