**Fall 2020, Math 579: Problem Set 4**
**Due: Thursday, September 24th, 2020**
**Binomial Theorem and Combinatorial Proofs**

**Discussion problems.** The problems below should be worked on in class.

(D1) *Combinatorial proofs.*

(a) Fill in the blanks in the following **combinatorial** proof that for any $n \geq 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

*Proof.* We will count the number of ways to choose subsets $A, B \subseteq [n]$ with $A \subseteq B$.

Right side: for each $i \in [n]$, either $i \in A$, $i \in B \setminus A$, or ____. This yields $3^n$ possibilities.

Left side: if we let $k = |B|$, then for each $k = 0, 1, \ldots, n$, there are ____ choices for $B$, and ____ ways to choose a subset $A \subseteq B$. This yields

$$2^0 \binom{n}{0} + 2^1 \binom{n}{1} + \cdots + 2^n \binom{n}{n}$$

possibilities. We conclude the left side must equal the right side. □

(b) Give a combinatorial proof that whenever $k \geq 0$ and $k + 3 \leq n$,

$$\binom{n}{3}\binom{n-3}{k} = \binom{n}{k}\binom{n-k}{3}.$$

(c) Give a combinatorial proof of the following identity for $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$

(D2) *The binomial theorem.* Recall the binomial theorem from Thursday:

$$(x+z)^n = \sum_{k=0}^{n} \binom{n}{k} x^k z^{n-k}.$$

(a) Use the binomial theorem to find the coefficient of $x^9 z^{15}$ in the expression $x^5(x^2 - z)^{17}$.

(b) Use the binomial theorem to prove that for any $n \geq 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

We call this an **algebraic** proof.

(c) Apply the binomial theorem (3 times!) to the expression

$$(x+1)^{n+1} = x(x+1)^n + (x+1)^n.$$

Then, reindex each sum to contain $x^{k+1}$ (as opposed to $x^k$) and pull out terms so that each sum starts at $k = 1$ and ends at $k = n$. Lastly, consolidate the right hand side into a single sum. Comparing coefficients on the left and right hand sides, what identity is obtained?

**Homework problems.** You must submit *all* homework problems in order to receive full credit.

(H1) Find the coefficient of $x^{11}z^7$ in the expansion of $(x+z)^{18} + x^3(x-z)^{15}$.

Hint: **don't expand**!!! This is what the binomial theorem is for!!!

(H2) Use induction on $n$ to prove that for all $n \geq 1$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: use the identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

in your inductive step.

(H3) Consider the following identity, valid for all $n \geq 1$:

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = 2^{n-2}n(n-1).$$

Give a **combinatorial** proof of this identity.

(H4) Give an **algebraic** proof that for $n \geq 1$,

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}.$$

Hint: write 3 and $-1$ each in a clever way, and then use the binomial theorem twice.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give a combinatorial proof of the identity

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n}{k-1}\binom{n-1}{k}\binom{n+1}{k+1}.$$