Fall 2020, Math 579: Problem Set 5
Due: Thursday, October 1st, 2020
Binomial Theorem (Deep Cuts) and Catalan Numbers

Discussion problems. The problems below should be worked on in class.

(D1) *The binomial theorem (deep cuts).* Recall the binomial theorem:

$$(x+z)^n = \sum_{k=0}^{n} \binom{n}{k} x^k z^{n-k}.$$

The goal of this problem is to prove for all $n \geq 1$,

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} = \frac{1}{n+1}.$$

(a) We can substitute into the binomial theorem to obtain

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Take the antiderivative of both sides of this equality with respect to x.
Note: don't make your Calculus teacher sad, be sure your answer has a $+C$ in it!

(b) The value of C in part (a) must be chosen so that equality holds for **all** values of x. Can we plug in a carefully chosen value of x to determine the value of C?

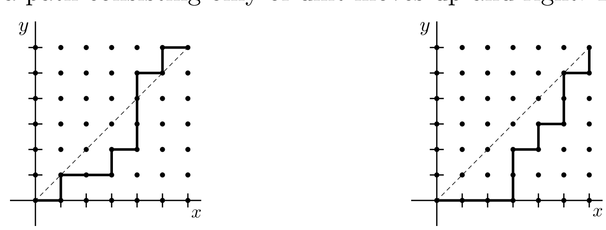
(c) Use part (b) to give an algebraic proof of the identity at the start of this problem.

(D2) *The Catalan numbers.* The following questions are an introduction to the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

a cornerstone of enumerative combinatorics.

(a) A *lattice path* is a path consisting only of unit moves up and right. For example:



Argue the number of lattice paths from $(0,0)$ to (m,n) for $m,n \in \mathbb{Z}_{\geq 0}$ equals $\binom{m+n}{n}$.

(b) A *Dyck path* is a lattice path between $(0,0)$ and (n,n) which does not pass above the line $y=x$. In the above examples, the right-hand lattice path is a Dyck path, but the left-hand lattice path is not. Draw all 5 Dyck paths between $(0,0)$ and $(3,3)$.

(c) Draw a lattice path $(0,0)$ to $(5,5)$ that is **not** Dyck path. Locate the first up move that occurs above the diagonal, and "flip" the rest of the path after that move (e.g., all up moves become right moves and visa versa). Do the same with a lattice path $(0,0)$ to $(7,7)$. What do you notice about the new ending points? Prove it.

(d) Prove that the number of Dyck paths between $(0,0)$ and (n,n) is given by C_n.

(e) It turns out that the Catalan numbers satisfy $C_0 = 1$ and $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ (this will be proven on the homework). Using this identity, give a combinatorial proof that the number of Dyck paths between $(0,0)$ and (n,n) is given by C_n.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Give a **combinatorial** proof that for all $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

(H2) Consider the following identity, valid for all $n \geq 1$:

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = 2^{n-2}n(n-1).$$

Give an **algebraic** proof of this identity (that is, using the binomial theorem).

(H3) Prove that the Catalan numbers satisfy $C_0 = 1$ and

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

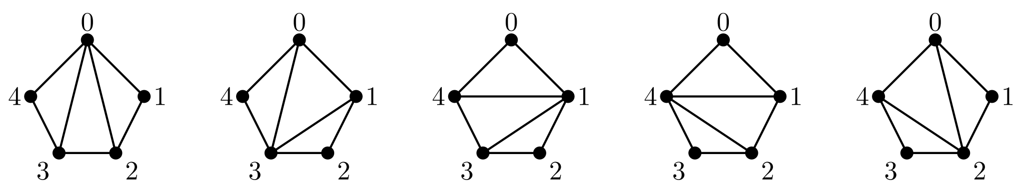
for all $n \geq 0$.

(H4) Determine the number of ways of properly parenthesizing the product of $n+1$ values. For example, if $n = 3$, then there are 5 ways:

$$((ab)c)d \qquad (ab)(cd) \qquad (a(bc))d \qquad a((bc)d) \qquad a(b(cd)).$$

Hint: you may find it helpful to find all 14 ways to do this when $n = 4$.

(H5) A *triangulation* of a regular n-gon (all sides are equal length and all interior angles are equal) is a way of drawing $n-3$ non-crossing diagonals to form $n-2$ triangles. For example, the following are triangluations of a regular pentagon.



Prove that the number of triangulations of a regular n-gon equals C_{n-2}.

Hint: you may do this bijectively (by finding a bijection between the triangulations of an n-gon and the Dyck paths from $(0,0)$ to $(n-2, n-2)$), or combinatorially (using the identity in Problem (H3) and a combinatorial proof), or by direct counting argument (using the formula for C_n in Problem (D2)).

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Problem (H5) gave 3 methods of completing the proof. Prove it again using 1 of the 2 methods you **did not** choose previously.