Fall 2020, Math 579: Problem Set 8
Due: Thursday, October 22nd, 2020
Recurrence Relations and Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Formal power series.*

(a) Fill in the blank in each of the following.

(i) $\displaystyle\sum_{n=0}^{\infty} \underline{\qquad} z^{\underline{\quad}} = \frac{2}{1-3z^2}.$

(ii) $\displaystyle\sum_{n=0}^{\infty} \underline{\qquad} z^{\underline{\quad}} = \frac{z}{2+z}.$

(b) Write each of the following as a rational function in z.

(i) $A(z) = \displaystyle\sum_{n=0}^{\infty} 2^{n-1} z^n.$

(ii) $A(z) = \displaystyle\sum_{n=0}^{\infty} (-1)^n 2^{2n} z^n.$

(D2) *Solving recurrence relations with generating functions.* For each of the following, (i) compute $a_0, \ldots, a_5$, (ii) use generating functions to find a formula for a_n in terms of n, and (iii) verify your formula for $n \le 5$.

(a) $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-2}$ for $n \ge 2$.

(b) $a_0 = 1$, $a_n = 2a_{n-1} + 1$ for $n \ge 1$.

(c) $a_0 = 3$, $a_1 = 1$, $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \ge 2$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use generating functions to find a_n if $a_0 = 1$ and $a_n = 3a_{n-1} + 2^n$ for $n \geq 1$.

(H2) Use generating functions to find L_n if $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Hint: we have $1 - z - z^2 = (1 - \omega z)(1 - \overline{\omega} z)$, where

$$\omega = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \overline{\omega} = \frac{1-\sqrt{5}}{2}.$$

(H3) Define $c(n) = 1$ if there exists a way to write n as a sum of the values 3 and 5, and $c(n) = 0$ otherwise. For instance, $c(13) = 1$ since $13 = 3 + 5 + 5$, and $c(15) = 1$ since $15 = 5 + 5 + 5$ (as well as $15 = 3 + 3 + 3 + 3 + 3$), but $c(7) = 0$ since there is no way to add 3's and 5's together to obtain 7. Prove that

$$\sum_{n=0}^{\infty} c(n) z^n = \frac{1 - z^{15}}{(1 - z^3)(1 - z^5)}.$$

Conjecture an analogous result if positive integers n_1 and n_2 replace 3 and 5. Note: you are **not** required to prove your conjecture.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove your conjecture from Problem (H3).