Fall 2020, Math 579: Problem Set 9
Due: Thursday, October 29th, 2020
Combinatorics of Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Integer partitions via generating functions.* Recall that $p(n)$ denotes the number of integer partitions of n (that is, the number of ways to write n as a sum of (unordered) positive integers), and that

$$P(z) = \sum_{n=0}^{\infty} p(n)z^n = \prod_{i=1}^{\infty} \frac{1}{1 - z^i}.$$

(a) Let $p_{\text{odd}}(n)$ denote the number of partitions of n with all odd parts. Show $p_{\text{odd}}(6) = 4$.

(b) Find an expression for $\sum_{n=0}^{\infty} p_{\text{odd}}(n)z^n$ by modifying the one for $P(z)$ above.

(c) Let $p_{\text{dis}}(n)$ denote the number of partitions of n in which no part appears more than once. Show $p_{\text{dis}}(6) = 4$.

(d) Find an expression for $\sum_{n=0}^{\infty} p_{\text{dis}}(n)z^n$ by modifying the one for $P(z)$ above.
Hint: how many times can "7" appear in a partition when all parts have to be distinct?

(e) Prove that $p_{\text{odd}}(n) = p_{\text{dis}}(n)$ for all $n \geq 0$ by proving that

$$\sum_{n\geq 0} p_{\text{odd}}(n)z^n = \sum_{n\geq 0} p_{\text{dis}}(n)z^n$$

using your expressions for the left and right hand sides in previous parts.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let m_n denote the number of ways to express $n \geq 0$ as a sum the integers 6, 9 and 20. For instance, $m_{18} = 2$ since $18 = 6 + 6 + 6 = 9 + 9$, and $m_{26} = 1$ since $26 = 6 + 20$ is the only way to write 26 as such a sum. Find a rational expression for the ordinary generating function of m_n. Do **not** attempt to find a formula for m_n!

(H2) For each of the following, find the an expression for the (ordinary) generating function $D(z) = \sum_{n=0}^{\infty} d_n z^n$ using the combinatorial interpretation of multiplication of power series. Note: **your final answer** to each part should be a **rational expression** for $D(z)$ in terms of z, and should not have any infinite sums or products. In particular, you are **not** required to find a formula for d_n!

(a) Suppose our course meets for n days. Let d_n denote the number of ways to split the available days into 3 units (counting methods, graph theory, and generating functions),

(i) select, from the first unit, some nonempty collection of days for a pop quiz,

(ii) select, from the second unit, an odd number of days to hold discussions, and

(iii) select, from the third unit, exactly 2 days in which to give an exam.

(b) What if in part (a), each unit should be at least 2 days long?

(c) What if in part (a), exactly half of the days in the first unit must have a pop quiz, and they must alternate days (either quiz, no quiz, quiz, no quiz, etc; or no quiz, quiz, no quiz, quiz, etc)?

(H3) Choose **one** of the parts of the previous problem, and use the generating function expression you obtained to find a formula for d_n. You may use a computer (e.g., `WolframAlpha`) to perform the partial fractions step for you.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) It is known that for each $d \geq 1$, we have

$$\sum_{n \geq 0} n^d z^n = \frac{Q_d(z)}{(1-z)^{d+1}}$$

for some polynomial $Q_d(z)$. Find a general formula, in terms of d, for the degree of $Q_d(z)$, the leading coefficient of $Q_d(z)$, and $Q_d(1)$ (that is, the sum of the coefficients of $Q_d(z)$).