Fall 2020, Math 579: Problem Set 10
Due: Thursday, November 5th, 2020
Composition of Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Combinatorics of generating functions.* Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$. Recall that the coefficients of $A(z)B(z)$ can be interpreted as the number of ways to

- split $[n]$ into two (possibly empty) subintervals $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$,
- place an "a-structure" on the first subinterval, and
- place a "b-structure" on the second subinterval.

Likewise, if $b_0 = 0$, the coefficients of $A(B(z))$ can be interpreted as the number of ways to

- split the set $[n]$ into some number of nonempty subintervals,
- place a b-structure on each subinterval, and
- place an a-structure on the set of subintervals.

(a) Let $A(z) = \sum_{n=1}^{\infty} z^n$. Using the fact that $A(z) = z/(1-z)$, find (algebraically) the coefficients of $A(A(z))$.

(b) Interpret the coefficients of $A(A(z))$ combinatorially. Verify this matches part (c).

(c) Let $B(z) = 1/(1-z)$. Interpret the coefficients of $A(B(z))$ combinatorially.
Note: this is not valid since $b_0 \neq 0$. Reconcile this with your answer.

(d) Let $c_n = n$ denote the number of ways to pick a person from a lineup of n people, and let $C(z) = \sum_{n=1}^{\infty} c_n z^n$ denote its generating function. With $A(z) = \sum_{n=1}^{\infty} z^n$ as above, describe what the coefficients of $C(A(z))$ represent, and find a formula using algebra.

(e) Interpret the coefficients of $A(C(z))$, then find a formula.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let

$$A(z) = \frac{z^2}{(1-z)^3} \qquad \text{and} \qquad B(z) = \frac{z}{(1-z)^2}.$$

Find a combinatorial interpretation of the coefficients of $A(B(z))$. You are **not** required to find a formula for the coefficients.

Note: your answer should be a word problem, e.g., involving a lineup of soldiers.

(H2) A permutation of $[n] = \{1, \ldots, n\}$ is called *indecomposable* if it cannot be split into a permutation on $\{1, \ldots, k\}$ and a permutation on $\{k+1, \ldots, n\}$ for $1 \leq k \leq n-1$. For example, 54321 is indecomposable, but 23154 can be decomposed as 231 and then 54.

Let c_n denote the number of indecomposable permutations on $[n]$, and let $c_0 = 0$. Find an equation relating the generating function $C(z) = \sum_{n=0}^{\infty} c_n z^n$ and the generating function for the number of permutations of $[n]$, that is, $P(z) = \sum_{n=0}^{\infty} (n!) z^n$.

(H3) Let

$$A(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \cdots$$

$$B(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \frac{1}{6!} z^6 + \cdots$$

(a) Show that $A'(z) = B(z)$ and $B'(z) = -A(z)$.

(b) Using part (a), show that $2A(z)A'(z) + 2B(z)B'(z) = 0$.

(c) Use part (b) to argue that $(A(z))^2 + (B(z))^2 = 1$. Thinking back to Calculus 2, what familiar identity does this equality encode?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) For each integer $t \geq 0$, consider the generating function

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} z^n.$$

Prove that $B_n(t)$ is a polynomial in t of degree n.