Fall 2020, Math 579: Problem Set 13
Due: Thursday, December 3rd, 2020
Graph Colorings

(D1) *Chromatic polynomials.* Fix a graph G with $n = |V(G)|$. The *chromatic function* of G is

$$\chi_G(k) = \#\text{ proper colorings of } G \text{ with } k \text{ colors.}$$

(a) Find the chromatic function $\chi_G(k)$, where G is each of the following graphs.

(i) The competely disconnected graph D_n with n vertices and no edges.

(ii) The complete graph K_n.

(b) Let c_i denote the number of ways to properly color G using **exactly** i colors. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \sum_{i=1}^{n} \binom{k}{i} c_i = \binom{k}{1} c_1 + \binom{k}{2} c_2 + \cdots + \binom{k}{n} c_n$$

for all $k \geq 1$. Use this to find $\chi_G(k)$ for $G = K_3$, and verify it matches part (a).

(c) Give a combinatorial proof of the identity in part (b).

(d) Using the fact that $\binom{k}{i} = \frac{1}{i!} k(k-1)\cdots(k-i+1)$ is a polynomial in k of degree i, deduce from part (b) that $\chi_G(k)$ is a polynomial in k of degree n.

(e) If a graph G has a loop, what can be said about $\chi_G(k)$?

(f) Fix an edge $e \in E(G)$. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \chi_{G \setminus e}(k) - \chi_{G/e}(k).$$

Using **only** this equality and part (a)(i), find the chromatic polynomial of $G = K_3$.

(g) Use the equality in part (e) to give an inductive proof that $\chi_G(k)$ is a polynomial. Does this proof guarantee that the coefficients of $\chi_G(k)$ are all integers?

Hint: use part (a)(i) as your base case.

(h) Give a combinatorial proof of the equality in part (f).

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the chromatic polynomial of each of the following graphs.

  (a) The graph G with 5 vertices $v_1, \ldots, v_5$ in which v_1 and v_5 both have v_2, v_3, and v_4 as neighbors (this yields a total of 6 edges).

  (b) The graph G obtained from K_n by removing one edge.

(H2) Fix a connected graph G with n vertices. Prove that G is a tree if and only if

$$\chi_G(k) = k(k-1)^{n-1}.$$

Hint: use Problem (H3) for the backwards direction.

(H3) Fix a graph G with n vertices and m edges.

  (a) Prove that the leading coefficient of $\chi_G(k)$ (that is, the coefficient of k^n) is 1.

  (b) Prove that the coefficient of k^{n-1} in the chromatic polynomial $\chi_G(k)$ equals $-m$.

Hint: each part of this problem can be proven in (at least) 2 distinct ways: one way uses a direct proof and the equation in Problem (D1)(b); another uses induction and the equation in Problem (D1)(f).

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Solve all parts of Problem (H3) using a **different** method than you already did.