Fall 2020, Math 579: Problem Set 15
Due: Friday, December 11th, 2020
Bipartite Graphs and Planar Graphs

Discussion problems. The problems below should be completed in class.

(D1) *Hall's Theorem.* Given a graph G and $T \subset V(G)$, let $N(T)$ denote the set of neighbors of vertices in T. In this problem, you will examine the following result in some special cases.

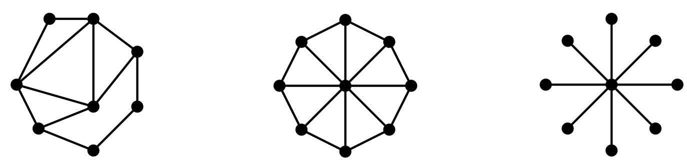
Theorem. *Fix a bipartite graph $G = (X, Y)$. There exists a perfect matching of X into Y if and only if $|T| \le |N(T)|$ for all $T \subset X$.*

(a) Verify Hall's Theorem holds in the 4 special cases when $|X| = 1$, $|X| = 2$, $|Y| = 1$ and $|Y| = 2$.

(b) Prove the forward direction of Hall's Theorem.

(D2) *Counting faces of planar graphs.* For a planar graph G, let V, E, and F denote the number of vertices, edges, and faces of G, respectively.

(a) Compute the quantity $V - E + F$ for each of the following graphs.



(b) Have each group member draw their favorite connected planar graph with at least 8 vertices and 15 edges, and compute $V - E + F$ for their graph.

(c) Notice this came out the same for each graph. This is known as *Euler's theorem* for planar, connected graphs. We will prove this by induction on E.

(i) Base case: prove Euler's theorem when $E = V - 1$. Why is this the base case?
(ii) **Carefully and precisely**, write the inductive hypothesis.
(iii) What can happen when an edge $e \in E(G)$ is removed?
(iv) Finish your proof that Euler's theorem holds for any planar graph G.

(D3) *Duals of planar graphs and a test for planarity.* For a planar graph G, let V, E, and F denote the number of vertices, edges, and faces of G, respectively.

(a) Use Euler's Theorem to give a non-pictorial proof that K_5 is not planar.
Hint: how many faces would it have, and how sides would each face need to have?

(b) Use Euler's Theorem to give a non-pictorial proof that $K_{3,3}$ is not planar.
Hint: is it possible for a face to have 3 boundary edges?

(c) Fix a simple (not necessarily planar!) graph G with V vertices and E edges.

(a) Prove that if G is planar, then $3F \le 2E$.
(b) Prove that if G is planar, then $E \le 3V - 6$.
(c) Is it true that any connected graph satisfying $E \le 3V - 6$ is planar?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Which of the following can be the degrees of the vertices of a bipartite graph?

(a) $3, 3, 3, 3, 3, 3$ (6 vertices total)

(b) $3, 3, 3, 3, 3, 3, 3, 3$ (8 vertices total)

(c) $3, 3, 3, 3, 3, 5, 6, 6, 6$ (9 vertices total)

(H2) Suppose G is a bipartite graph in which every vertex has degree k. Prove that G has a perfect matching.

(H3) OMITTED

(H4) Suppose for a given planar graph G, each face of G (including the "outside" one!) has either 3 or 5 boundary edges. Prove that the number of faces of G is even.

(H5) Prove that if any 2 edges are removed from the graph K_6, the result is not planar. Is the same true if we remove 3 edges?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $n \geq 1$ and $k < n/2$. Let $G = (X, Y)$ denote a bipartite graph so that X consists of the k-element subsets of $[n]$, Y consists of the $(k+1)$-element subsets of $[n]$, and $(x, y) \in E(G)$ for $x \in X$ and $y \in Y$ precisely when $x \subset y$. Find a perfect matching for X into Y.