Fall 2020, Math 620: Week 1 Problem Set
Due: Thursday, September 3rd, 2020
Introduction to Groups

Discussion problems. The problems below should be completed in class.

(D1) *Checking group axioms.* Determine which (if any) of the group axioms are violated by each of the following sets G under the given operation $*$. **Brief** justifications are sufficient for this problem (no formal proof is required).

(a) $G = \mathbb{Z}$; $a * b = a - b$.

(b) $G = \mathbb{Z}_{\geq 0}$; $a * b = a + b$.

(c) $G = \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).

(d) $G = \{1, 3, 7, 9\} \subset \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).

(e) $G = \mathbb{R} \times \mathbb{R}$; $(a, b) * (c, d) = (ac, bd)$.

(D2) *Group element orders.* For a group $(G, \cdot)$ and $a \in G$, the *order* of a, denoted $|a|$, is the smallest $n \in \mathbb{Z}_{\geq 1}$ such that $a^n = e$, or $|a| = \infty$ is no such k exists.

(a) Find the order of each element of D_4. Do the same for $\mathbb{Z}$.

(b) Fix a group $(G, \cdot)$ and an element $a \in G$ with $|a| = n$. Fill in the gaps in the following proof that $a^k = e$ if and only if $n \mid k$ (you may insert extra blanks if you prefer).

Proof. Suppose $a^k = e$, and write $k = qn + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < n$. Then

$$\begin{aligned} e = a^k &= a^{qn+r} \\ &= \underline{\hspace{3cm}} \\ &= \underline{\hspace{3cm}} \\ &= a^r, \end{aligned}$$

so by the minimality of n, we must have $r = 0$. As such, $n \mid k$. □

(c) Prove that if G is finite and $a \in G$, then $|a|$ is finite.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Determine whether each of the following sets G form a group under the given operation $*$. Prove your assertions.

(a) $G = \{1, 3, 5, 7, 9\} \subset \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).

(b) $G = \mathbb{R}$; $a * b = a + b + 3$.

(c) G is the set of nonzero real numbers; $a * b = |a| \cdot b$.

(H2) Suppose G is a group and $a, b, c \in G$. Prove each of the following statements Using **only group axioms**, prove that if $ab = ac$, then $b = c$. Be especially careful with associativity! In particular, any triple products xyz should be written as either $(xy)z$ or $x(yz)$.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If $(G, \cdot)$ is a group and $a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$.

(b) If $(G, \cdot)$ is a group and $a, b, \in G$ with $|a| = n$ and $|b| = m$, then $|ab| \leq \operatorname{lcm}(n, m)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose $(G, *)$ is a group, where $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $*$ is an operation satisfying

(i) $a * b \leq a + b$ for every $a, b \in G$, and

(ii) $a * a = 0$ for every $a \in G$.

Write out the operation table for G, and **briefly** justify why this is the only possibility.