Fall 2020, Math 620: Week 2 Problem Set
Due: Wednesday, September 9th, 2020
Subgroups, Cyclic Groups, and Isomorphisms

Discussion problems. The problems below should be completed in class.

(D1) *Cyclic groups.* A group $(G, \cdot)$ is *cyclic* if it can be generated by a single element (that is, for some $a \in G$, we have $G = \langle a \rangle = \{a^r : r \in \mathbb{Z}\}$.)

(a) Suppose $n = |G|$ is finite. Complete the following proof that $G \cong (\mathbb{Z}_n, +)$.

Proof. Since G is cyclic, there is some $a \in G$ with $G = \{a^r : r \in \mathbb{Z}\}$. Consider the map

$$\begin{aligned} \varphi : G &\longrightarrow \mathbb{Z}_n \\ a^r &\longmapsto [r]_n. \end{aligned}$$

First, we see φ is well-defined (that is, for any $r, t \in \mathbb{Z}$, $a^r = a^t$ implies $\varphi(a^r) = \varphi(a^t)$). Moreover, φ is an isomorphism. This completes the proof. $\square$

(b) Give a sketch of a proof that if G is infinite and cyclic then $G \cong \mathbb{Z}$.

(c) Conclude that up to isomorphism, the only cyclic groups are $\mathbb{Z}_n$ for $n \geq 1$ and $\mathbb{Z}$.

(D2) *Group automorphisms.* An *automorphism* of a group G is an isomorphism $G \to G$. The set of automporphisms of G, denoted $\operatorname{Aut}(G)$, is itself a group under composition.

(a) Find all elements of $\operatorname{Aut}(\mathbb{Z})$. Hint: where can an automorphism send $1 \in \mathbb{Z}$?

(b) Find $\operatorname{Aut}(\mathbb{Z}_6)$, $\operatorname{Aut}(\mathbb{Z}_7)$, and $\operatorname{Aut}(\mathbb{Z}_8)$.

(c) State and prove a characterization of $|\operatorname{Aut}(\mathbb{Z}_n)|$.

(d) Find all elements of $\operatorname{Aut}(\mathbb{Z}_2^3)$.

(D3) *Working with permutations.* Consider the following permutations.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 6 & 3 & 8 & 2 & 5 & 7 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 8 & 6 & 1 & 4 & 7 & 3 \end{pmatrix}$$

(a) Write each of the above permutations as a product of disjoint cycles.

(b) Find the order of each element above. Hint: do the disjoint cycles help?

(c) Write each of the above permutations as a product of 2-cycles in two different ways.

(d) Determine which of the above permutations are even, and which are odd.

(e) Is it possible to write σ or τ as a product of disjoint 2-cycles?

(f) Write σ^{-1}, τ^{-1}, and α^{-1} as a products of disjoint cycles, and as products of 2-cycles. Hint: you have already written σ, τ, and α in these forms!

(g) Determine whether σ^{91} is even or odd.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Identify a subgroup of $GL_2(\mathbb{R})$ isomorphic to D_4. Identify a subgroup isomorphic to $\mathbb{Z}_6$.

(H2) It turns out $GL_2(\mathbb{Z}_2)$ is isomophic to either S_3 or $\mathbb{Z}_6$. Determine which.

(H3) Identify an element of S_9 of order 20.

(H4) Let $B_n \subset S_n$ denote the set of odd permutations in S_n. Prove that $|B_n| = |A_n| = n!/2$. Hint: can you find a bijection $f : A_n \to B_n$?

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) The group $(\mathbb{Q}, +)$ is cyclic.

(b) If G is a group and every proper subgroup of G is cyclic, then G is cyclic.

(c) For any groups G_1 and G_2, we have $\mathrm{Aut}(G_1 \times G_2) \cong \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) A *graph* H is a collection E of 2-element subsets of $\{1, \ldots, n\}$ (called *edges*). An *automorphism* of a graph is a permutation σ of the integers $1, \ldots, n$ such that $\{a, b\} \in E$ if and only if $\{\sigma(a), \sigma(b)\} \in E$. The set $\mathrm{Aut}(H)$ of automorphisms of a graph H is a group under composition (you are not required to prove this). For example, if H is the 4-cycle graph, with edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, and $\{4, 1\}$, then $\mathrm{Aut}(H) \cong D_4$.

Identify a graph H whose automorphism group $\mathrm{Aut}(H)$ is isomorphic to $(\mathbb{Z}_5, +)$.