Fall 2020, Math 620: Week 3 Problem Set
Due: Wednesday, September 16th, 2020
Permutation Groups

Discussion problems. The problems below should be completed in class.

(D1) *Group elements as permutations.* The goal of this problem is to develop a proof of the following theorem, as well as intuition for why it holds.

Theorem. *Every finite group G with $|G| = n$ is isomorphic to a subgroup of S_n.*

(a) The goal of the first few parts is to identify a subgroup of S_6 isomorphic to D_3. Choose a labeling of the elements of D_3 as $a_1, a_2, \ldots, a_6$ (you may do this in any way you wish).

(b) Let $r \in D_3$ denote clockwise rotation by $120°$. Define a permutation $\sigma \in S_6$ given by $\sigma(i) = k$ where $a_k = ra_i$. Verify that σ is indeed a permutation by writing it in permutation notation.

(c) Repeat the previous part for each $a \in D_3$ (that is, define a permutation $\sigma_a \in S_6$ given by $\sigma(i) = k$ where $a_k = aa_i$). Write down all 6 resulting permutations (you may want to "divide and conquer" amongst your groupmates to save time!).

(d) Verify in three examples that for any $a, b \in D_3$, the permutation corresponding to ab equals the product of the permutations corresponding to a and b.

(e) Using the ideas above, find a subgroup of S_4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove S_n is isomorphic to a subgroup of A_{n+2}.

(H2) Locate a generating set for S_n consisting of only 2 generators.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) For each $n \geq 3$, every permutation in S_n can be written as a product of 3-cycles.
(b) For each $n \geq 2$, every permutation in S_n is a product of at most $n-1$ transpositions.
(c) For each $n \geq 3$, every permutation in S_n is a product of adjacent transpositions.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine what familiar group is isomorphic to $\mathrm{Aut}(S_3)$.