Fall 2020, Math 620: Week 4 Problem Set
Due: Thursday, September 24th, 2020
Cosets and Quotient Groups

Discussion problems. The problems below should be completed in class.

(D1) *Arithmetic in quotient groups.* Consider the unit interval as an additive group, viewed as the quotient group $\mathbb{R}/\mathbb{Z}$.

(a) Explain why every element of $\mathbb{R}/\mathbb{Z}$ has a representative in $[0, 1)$.

(b) Show that $\mathbb{R}/\mathbb{Z}$ has elements of every possible order.

(c) Does $\mathbb{Q}/\mathbb{Z}$ have elements of every possible order?

(d) Does $\mathbb{Q}/\mathbb{Z}$ have infinitely many elements of a particular order?

(D2) *Subgroup correspondence.* The goal of this problem is to discover the "correspondence theorem" for subgroups of quotient groups.

(a) Draw the subgroup lattice of $\mathbb{Z}_{24}$. Write each subgroup in the form $\langle n \rangle$.

(b) Do the same for the subgroup lattice of $\mathbb{Z}_{72}$.

(c) Locate a subgroup $H \subset \mathbb{Z}_{72}$ so that $\mathbb{Z}_{72}/H \cong \mathbb{Z}_{24}$. Do you notice any relationship between the subgroup lattice of $\mathbb{Z}_{24}$ and the subgroups of $\mathbb{Z}_{72}$ containing H?

(d) If there are more than 10 minutes left in class, find all subgroups of the group D_4. Use your list to determine how many subgroups the quotient group $D_4/\{e, r^2\}$ has.

(e) Fix a group G and a normal subgroup $H \lhd G$. Formulate a conjecture specifying a bijection between the subgroups of G/H and certain subgroups of G. Use the map $G \to G/H$ to specify the bijection.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose G is a group and $H \subset G$ is a subgroup. Prove **without using any theorems** that $b \in aH$ if and only if $bH = aH$.

(H2) Prove A_n is a normal subgroup of S_n.

(H3) Fix a group G. Define the *center* of G as the set

$$C = \{c \in G : ca = ac \text{ for all } a \in G\}$$

of elements that commute with every element of G.

(a) Prove C is a normal subgroup of G.

(b) Prove or disprove: G/C has trivial center.

(H4) Fix a group $(G, \cdot)$ and a normal subgroup $H \subset G$, and consider the map

$$\begin{aligned} \varphi : G &\longrightarrow G/H \\ a &\longmapsto [a]_H. \end{aligned}$$

The goal of this problem is to prove the correspondence theorem from Problem (D2).

(a) Prove that if $K \subseteq G/H$ is a subgroup, then $\varphi^{-1}(K)$ is a subgroup of G containing H.

(b) Prove that if $K \subseteq G$ is a subgroup with $H \subseteq K$, then $\varphi(K)$ is a subgroup of G/H.

(H5) The following questions pertain to the correspondence theorem from Problem (D2).

(a) If a subgroup of G/H is normal, is its corresponding subgroup of G normal? What about the converse of this statement?

(b) If a subgroup of G/H is cyclic, is its corresponding subgroup of G cyclic? What about the converse of this statement?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose p is prime. Find all groups G (up to isomorphism) with $|G| = p^2$.