Fall 2020, Math 620: Week 5 Problem Set
Due: Thursday, October 1st, 2020
Isomorphism Theorems

Discussion problems. The problems below should be worked on in class.

(D1) *The first isomorphism theorem.* Prove the following using the first isomorphism theorem.

(a) For any groups G and G', $G \times G'/(G \times \{e\}) \cong G'$. (Here, $G \times \{e\}$ is shorthand for the subgroup $\{(g, e) : g \in G\} \subset G \times G'$).

(b) $D_{24}/\langle r^6 \rangle \cong D_6$.

(D2) *The third isomorphism theorem.* The goal of this problem is to prove the following theorem. Tip: due to the large number of quotients in this problem, **use the bracket notation** for all quotient group elements (e.g., $[a]_H$).

Theorem. *If $(G, \cdot)$ is a group and $K, H \lhd G$ with $K \subset H$, then $(G/K)/(H/K) \cong G/H$.*

(a) Let $G = \mathbb{Z}_{24}$, $H = \{[3k]_{24} : k \in \mathbb{Z}\}$, and $K = \{[12k]_{24} : k \in \mathbb{Z}\}$. Verify the above theorem holds in this case.

(b) Explain why H/K is a subset of G/K. Use the word "coset" in your explanation.

(c) Prove the third isomorphism theorem using the first isomorphism theorem. In particular, find a homomorphism $\phi : G/K \to G/H$ whose kernel equals H/K.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that if $H \subseteq G$ is a subgroup and $[G : H] = 2$, then H is normal.

(H2) Suppose G is a group. Given $a \in G$, define $f_a : G \to G$ by $f_a(x) = axa^{-1}$.

(a) Prove f_a is an automorphism (these are known as *inner automorphisms*).

(b) Let $G' = \{f_a : a \in G\} \subset \mathrm{Aut}(G)$. Prove G' is a normal subgroup of $\mathrm{Aut}(G)$.

(c) Let $\varphi : G \to G'$ denote the map $a \mapsto f_a$. Characterize the elements of G in $\ker(\varphi)$.

(d) Characterize which groups G have a unique inner automorphism.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If G is a group and $H, K \lhd G$ with $K \subset H$, then $G/H \times H/K \cong G/K$.

(b) If G, G' are groups and $H \lhd G$, $H' \lhd G'$, then $(G \times G')/(H \times H') \cong (G/H) \times (G'/H')$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose G is a group and $H, K \lhd G$ with $HK = G$. Determine under what condition(s) involving H and K we have $G \cong G/H \times G/K$.