Fall 2020, Math 620: Week 6 Problem Set
Due: Thursday, October 8th, 2020
Relations and Group Presentations

Discussion problems. The problems below should be completed in class.

(D1) *Free groups and relations.*

(a) Let $F_1 = \langle a \rangle$ denote the free group on 1 generator. Locate an isomorphism $F_1 \to \mathbb{Z}$. (You do not have to prove it is an isomorphism.)

(b) Draw the Cayley graph of the group $G = \langle r, f \mid f^2, rfrf \rangle$ (this is identical to the presentation of D_n without the r^n relation). Is G finite or infinite?

(c) Let $G = \langle a, b \mid a^2, b^2, aba^{-1}b^{-1} \rangle$. Determine what more familiar group G is.

(D2) *Group presentations.* Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators.

(a) For each of the following normal subgroups H_1 and H_2, prove $H_1 = H_2$ by representing each generator of H_1 as a product of conjugates of generators of H_2, and visa versa.

(i) $H_1 = \langle a^3b^2 \rangle$, $H_2 = \langle ba^3b \rangle$.

(ii) $H_1 = \langle aba^{-1}b^{-1}, a^2b^3ab^4 \rangle$, $H_2 = \langle aba^{-1}b^{-1}, a^3b^7 \rangle$.

(iii) $H_1 = \langle a^4ba^{-3}b^{-2}, a^3b^2a^{-2}b^{-3}, a^2b^3a^{-1}b^{-4} \rangle$,
$H_2 = \langle a^4ba^{-3}b^{-2}, a^3b^2a^{-2}b^{-3}, a^4ba^{-1}b^{-4} \rangle$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let $G = \langle a, b \mid aba^{-1}b^{-2}, bab^{-1}a^{-2} \rangle$. Determine $|G|$.

(H2) Identify which "familiar" group each of the following is isomorphic to.

(a) $G = \langle a, b \mid a^4, b^2, aba^{-1}b^{-1} \rangle$

(b) $G = \langle a, b, c \mid a^4, b^2, c^2, abc, acac \rangle$

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If $F_2 = \langle a, b \rangle$ is the free group on 2 generators, then the **subgroup** $H = \langle ab, ab^2 \rangle \subset F_2$ and **normal subgroup** $H' = \langle ab, ab^2 \rangle \subset F_2$ are identical subsets of F_2.

(b) The group $(\mathbb{Q}, +)$ is a quotient of the free group on 2 generators.

(c) Up to isomorphism, the only non-Abelian group of order 8 is D_4.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine which familiar group is isomorphic to $G = \langle a, b \mid a^4, b^2, (ab)^3 \rangle$.