Fall 2020, Math 620: Week 8 Problem Set
Due: Thursday, October 22nd, 2020
Ideals and Quotient Rings

Discussion problems. The problems below should be worked on in class.

(D1) *Polynomial ring quotients.*

(a) What is a single element that generates the ideal $I = \langle x^2 - 2x, x^3 + 5x^2 \rangle \subset \mathbb{Q}[x]$?

(b) In the ring $R = \mathbb{R}[x]/\langle x^2 + 1 \rangle$, prove $[x]^2 = [-1]$. Prove $R = \{[ax + b] : a, b \in \mathbb{R}\}$.

(c) Let $\mathbb{Q}[\sqrt{2}] = \{f(\sqrt{2}) : f(x) \in \mathbb{Q}[x]\}$. What does a general element of $\mathbb{Q}[\sqrt{2}]$ look like? Write a general multiplication rule for $\mathbb{Q}[\sqrt{2}]$.

(d) Identify $\mathbb{Q}[\sqrt{2}]$ as a quotient of the polynomial ring $\mathbb{Q}[x]$.

(e) Determine whether $\mathbb{R}[x]/\langle x^2 + 2 \rangle$ is an integral domain, a field, or neither.

(f) Suppose R is a ring such that some $t \in R \setminus \{1, -1\}$ has $t^2 = 1$. Prove R is not a field. Find an example of such a ring.

(D2) *Prime and maximal ideals.* Fix a commutative ring R and an ideal I. We say I is *prime* if R/I is an integral domain, and I is *maximal* if R/I is a field.

(a) Suppose R is a field. Classify all ideals of R.

(b) Suppose $I \subset R$ is maximal, and fix an ideal J with $I \subseteq J \subsetneq R$. Based on part (a), what can be said about J?

(c) State and prove an equivalent condition for I to be maximal that avoids quotient rings.

(d) Suppose I is prime, and fix ring elements $a, b \notin I$. Under what conditions is $ab \in I$?

(e) State and prove an equivalent condition for I to be prime that avoids quotient rings.

(f) Identify a ring R and a nonzero ideal $I \subset R$ that is prime but not maximal.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider the rings $R_1 = \mathbb{Q}[x]/\langle x^2 - 1\rangle$ and $R_2 = \mathbb{Q}[x]/\langle x^2 + 1\rangle$.

- (a) Are R_1 and R_2 isomorphic as additive groups?
- (b) Are R_1 and R_2 isomorphic as rings?

(H2) Consider the ring

$$\mathbb{Q}[\![x]\!] = \{a_0 + a_1 x + a_2 x^2 + \cdots : a_i \in \mathbb{Q}\}$$

of *formal power series* with coefficients in $\mathbb{Q}$. Here, addition is term-by-term, and multiplication is given by distribution using the following formula.

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + \cdots) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

Notice that on the right side above, each coefficient involves only a **finite** sum! This way we avoid any questions of convergence.

- (a) Determine which elements of $\mathbb{Q}[\![x]\!]$ are units.
- (b) Find all maximal ideals of $\mathbb{Q}[\![x]\!]$.
- (c) Prove $\mathbb{Q}[\![x]\!]/\langle x^2\rangle \cong \mathbb{Q}[x]/\langle x^2\rangle$.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

- (a) Every nonzero element of $R = \mathbb{Q}[x]/\langle x^2\rangle$ is either a unit or a zero-divisor.
- (b) The characteristic of any field is either 0 or prime.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: any integral domain R with finitely many elements is a field.