Fall 2020, Math 620: Week 9 Problem Set
Due: Thursday, October 29th, 2020
Rings of Fractions and Localization

Discussion problems. The problems below should be completed in class.

(D1) *Constructing the rationals from the integers.* Define an equivalence relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}_{\neq 0}$ by $(a, b) \sim (a', b')$ when $ab' = a'b$. Let Q denote the set of equivalence classes of $\sim$. Define operations $\oplus$ and $\odot$ on Q so that

$$[(a, b)] \oplus [(c, d)] = [(ad + bc, bd)] \qquad \text{and} \qquad [(a, b)] \odot [(c, d)] = [(ac, bd)]$$

for all $[(a, b)], [(c, d)] \in Q$. (Intiutively, $[(a, b)] \in Q$ represents $a/b \in \mathbb{Q}$.)

(a) Prove addition is well-defined in Q, that is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(a, b)] \oplus [(c, d)] = [(a', b')] \oplus [(c', d')]$. Do the same for multiplication.

(b) Prove addition in Q is commutative and associative.

(c) Prove multiplication in Q is commutative and associative.

(d) Prove distributivity holds in Q.

(e) Prove every element of Q has an additive inverse, and that every nonzero element has a multiplicative inverse.

(D2) *Localization.*

(a) Let $P = \langle 5 \rangle \subset \mathbb{Z}$ and $D = \mathbb{Z} \setminus P$ (that is, the set complement of P in $\mathbb{Z}$). Verify that D is a multiplicative set.

(b) Let $I = \langle 6 \rangle \subset \mathbb{Z}$ and $D = \mathbb{Z} \setminus I$. Is D a multiplicative set?

(c) Determine for which ideals $J \subset \mathbb{Z}$ the set $\mathbb{Z} \setminus J$ is a multiplicative set.

(d) Let $P = \langle 5 \rangle \subset \mathbb{Z}$ and $D = \mathbb{Z} \setminus P$. Find all ideals of $D^{-1}\mathbb{Z}$.

Hint: you may use that every ideal in $\mathbb{Z}$ has the form $\langle a \rangle$ for some $a \in \mathbb{Z}$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Write up a proof for any one part of Problem (D1) you choose. You may use "free of charge" (i.e., without proof) any prior parts of the same discussion problem.

(H2) Fix a field F with $\operatorname{char} F = 0$. Prove that F has a subring isomorphic to $\mathbb{Q}$.

(H3) Fix a ring R and a prime ideal $P \subset R$, and let $D = R \setminus P$.

(a) Prove that D is a multiplicative set.

(b) Prove that $D^{-1}R$ has a unique maximal ideal (the ring $R_P = D^{-1}R$ is called the *localization of* R *at* P and plays an important role in algebraic geometry).

(c) Classify the ideals of R_P in terms of the ideals of R.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If $P \subset R$ is a maximal ideal, then R_P has a unique proper nontrivial ideal.

(b) In an integral domain, every ideal is prime.

(c) For all $a, b, c \in \mathbb{Q}$ with $a \neq 0$, we have $\mathbb{Q}[x, y]/\langle ax + by + c\rangle \cong \mathbb{Q}[t]$.