Fall 2020, Math 620: Week 13 Problem Set
Due: Thursday, December 3rd, 2020
Field Extensions

Discussion problems. The problems below should be completed in class.

(D1) *Splitting fields.*

(a) Find the minimal polynomials of $\sqrt{2}$, $3\sqrt{2}+4$, and $\sqrt[3]{2}+1$ over $\mathbb{Q}$.
(b) Argue that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
(c) Find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$.
(d) Does $\mathbb{Q}(\sqrt{m}, \sqrt{n})$ have the same degree for any distinct integers $m, n \ge 2$?
(e) Find the splitting field of $(x^2-5)(x^2-7)$ over $\mathbb{Q}$.
(f) Find the splitting field of x^4-4 over $\mathbb{Q}$.
(g) Find the splitting field of x^d-1 over $\mathbb{Q}$ for $d = 2$, 3, 4, 5, and 6.
(h) Conjecture a formula in $d \ge 2$ for the degree of the splitting field of x^d-1 over $\mathbb{Q}$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the minimal polynomial of $\sqrt{3} + \sqrt[3]{2}$ over $\mathbb{Q}$.

(H2) Find the splitting field of $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ over $\mathbb{Z}_3$.

(H3) Fix a prime p. In this problem, we will construct the algebraic closure of $\mathbb{Z}_p$. Hint: you may find The Key Lemma useful frequently!

(a) Fix $t \geq 1$. Prove $\mathbb{F}_{p^t}$ has a subfield isomorphic to $\mathbb{F}_{p^r}$ if and only if $r \mid t$.

(b) Prove that if $r \mid t$, then there is a **unique** subfield of $\mathbb{F}_{p^t}$ isomorphic to $\mathbb{F}_{p^r}$.
In light of this, in what follows, when $r \mid t$, it is natural to write $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^t}$, identifying $\mathbb{F}_{p^r}$ with the subfield of $\mathbb{F}_{p^t}$ it is isomorphic to.

(c) Let $F = \bigcup_{t \geq 1} \mathbb{F}_{p^t}$. Prove that F is a field.

(d) Prove that F is algebraically closed.

(e) Prove that $F = \overline{\mathbb{Z}}_p$, i.e., there is no algebraically closed field F' with $\mathbb{F}_q \subseteq F' \subsetneq F$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a field F that has cardinality strictly larger than $\mathbb{R}$.