Fall 2021, Math 596: Week 1 Problem Set
Due: Thursday, September 2nd, 2021
Preview of Geometry

Discussion problems. The problems below should be worked on in class.

(D1) *Exhibiting Ehrhart's theorem.* For each polytope P below, do the following:

(i) draw P (as best you can);
(ii) find $L_P(t)$ for $t = 1, 2, 3, 4$;
(iii) use the values in (ii) to find the coefficients of the Ehrhart polynomial of P; and
(iv) verify that the constant term and leading coefficient of $L_P(t)$ are as predicted by Ehrhart's theorem.

Please verify your answers for part (a) with me before continuing on to part (b).

(a) $P = \text{conv}\{(0,1), (1,0), (-1,0), (-1,1), (-1,-1)\}$.
(b) $P = \text{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$.
(c) $P = \text{conv}\{(1,0,0), (0,1,0), (0,0,1)\}$.

(D2) *Pick's theorem.*

(a) Locate a lattice triangle $T \subset \mathbb{R}^2$ (that is, a triangle whose vertices are lattice points) such that T contains exactly 3 lattice points (namely, it's vertices) and has the largest possible area.
(b) Locate a lattice triangle T containing exactly 4 lattice points, one of which lies on the interior of T, and has the largest possible area.
(c) Suppose $P \subset \mathbb{R}^2$ is a lattice polygon, and let I, B, and A denote the number of interior lattice points of P, boundary lattice points of P, and area of P, respectively. By Ehrhart's theorem, we know

$$L_P(t) = at^2 + bt + c$$

for some $a, b, c \in \mathbb{Q}$. Find a formula for a, b, and c in terms of A, B, and I.
(d) Use Ehrhart reciprocity to obtain an equation relating A, B, and I. This is known as *Pick's theorem.* Note: your answer here should **not** involve t!
(e) For each $m \in \mathbb{Z}_{\geq 1}$, consider the polytope

$$P = \text{conv}\{(1,0,0), (0,1,0), (1,1,0), (0,0,m)\}.$$

Find the volume A of P, the number I of interior points of P, and the number B of boundary points of P (note that some, but not all, of these will depend on m). State, in one sentence, what this tells us about the possibility of generalizing Pick's theorem to 3 dimensions.

Hint: you may look up the formula for the volume of a pyramid with triangular base.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the Ehrhart polynomial of

$$P = \text{conv}\{(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,0,0),\ (1,1,0)\},$$

which is a 3D cube with 2 adjacent vertices removed.

(H2) Let $P = [0, \frac{1}{2}]$ (that is, the closed interval from 0 to $\frac{1}{2}$). Find a formula for $L_P(t)$, and prove your formula holds.

Hint: find $L_P(t)$ for $t = 1, 2, 3, \ldots$ until you are convinced you have found a pattern.

(H3) Let $P = \text{conv}\{(0,0),\ (0,1),\ (\frac{1}{2}, 0)\}$. Find a formula for $L_P(t)$, and prove your formula holds.

Hint: your formula will likely have cases based on whether t is even or odd.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If P is a polygon whose vertices lie in $\mathbb{Q}^2$, then Pick's theorem holds for P (that is,

$$A = I + \tfrac{1}{2}B - 1$$

where A, B, and I denote the area of P, the number of boundary lattice points of P, and the number of interior lattice points of P, respectively).

(b) Any polytope P satisfies $L_P(t) > 0$ for some $t \geq 1$.