Fall 2021, Math 596: Week 2 Problem Set
Due: Thursday, September 9th, 2021
Introduction to Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Manipulating power series.* In what follows, denote

$$A(z) = \sum_{n\geq 0} a_n z^n \qquad \text{and} \qquad B(z) = \sum_{n\geq 0} b_n z^n.$$

(a) For each of the following, find an expression for $A(z)$ that has no sigma-sums. You may find it helpful to reference the first centered equation in Problem (D2), which was also seen in lecture.

(i) $a_n = 4n + 3$

(ii) $a_n = 2n + 1$ if n is even, $a_n = 3n - 1$ if n is odd.

(b) Find a formula for a_n in terms of n given the following $A(z)$.

(i) $A(z) = \dfrac{3z + 17}{1 - z^2}$.

(ii) $A(z) = \dfrac{3z + 17}{(1 - z)^2}$.

(c) For each of the following $A(z)$, find a formula for the coefficients of $B(z) = 1/A(z)$. Then, prove via direct multiplication that $A(z)B(z) = 1$.

(i) $A(z) = 1 - 3z + 2z^2$
Hint: find the first few coefficients of $B(z)$ by hand, conjecture a formula for b_n.

(ii) $A(z) = \sum_{n\geq 0}(-1)^n z^n$
Hint: start by writing $A(z)$ as a rational function.

(D2) *Power series of polynomial functions.* Recall that in lecture, we saw

$$1 + 2z + 3z^2 + \cdots = \sum_{n=0}^{\infty}(n + 1)z^n = \frac{1}{(1 - z)^2},$$

and that the "formal derivative" of $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is

$$A'(z) = \frac{d}{dz}A(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=0}^{\infty}(n + 1)a_{n+1}z^n.$$

(a) Manipulate the first expression to write $\sum_{n=0}^{\infty} nz^n$ as a rational expression in z.

(b) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^2 z^n$ as a rational expression in z.

(c) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^3 z^n$ as a rational expression in z.

(d) Give a proof by induction that for each $d \geq 1$,

$$\sum_{n=0}^{\infty} n^d z^n = \frac{Q_d(z)}{(1 - z)^{d+1}}$$

for some polynomial $Q_d(z)$.
Hint: start by identifying $Q_1(z)$, $Q_2(z)$, and $Q_3(z)$ from parts (a), (b), and (c).

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find a rational expression for $A(z) = \sum_{\geq 0} a_n z^n$ in terms of z for each of the following sequences a_n. Your answer should **not** contain any "$\cdots$" or sigma-sums.

Hint: you can substantially shorten the algebra needed to obtain your answer with clever use of power series we have already encountered.

(a) $a_n = n^3 + 7n^2 + 3n - 5$

(b) $a_n = n2^n - n^2 3^n$

(H2) Find a formula for a_n for each of the following power series $A(z) = \sum_{n\geq 0} a_n z^n$.

Note: your formula for a_n should be in terms of n and have no sigma-sums in it!

(a) $A(z) = \dfrac{1}{2+z}$

(b) $A(z) = \dfrac{5+2z+z^2}{1-z^3}$

(c) $A(z) = \dfrac{z^2+1}{(1-z)^3}$

Hint: start by writing this series as a combination of those from Problem (D2).

(H3) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$. Prove that $A(z)$ has a multiplicative inverse if and only if $a_0 \neq 0$. Hint: if $A(z)B(z) = 1$ for some $B(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, show each b_n is determined by $a_0, a_1, \ldots, a_n$.

(H4) Define $c(n) = 1$ if there exists a way to write n as a sum of the values 3 and 5, and $c(n) = 0$ otherwise. For instance, $c(13) = 1$ since $13 = 3+5+5$, and $c(15) = 1$ since $15 = 5+5+5$ (as well as $15 = 3+3+3+3+3$), but $c(7) = 0$ since there is no way to add 3's and 5's together to obtain 7. Prove that

$$\sum_{n=0}^{\infty} c(n) z^n = \frac{1-z^{15}}{(1-z^3)(1-z^5)}.$$

Note: $c(0) = 1$ since $0 = 0 \cdot 3 + 0 \cdot 5$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Conjecture and prove an analogous result to Problem (H4) where positive integers n_1 and n_2 replace 3 and 5.