Fall 2021, Math 596: Week 3 Problem Set
Due: Thursday, September 16th, 2021
Generating Functions - Deep Cuts

Discussion problems. The problems below should be worked on in class.

(D1) *Recurrence relations.* For each of the following, (i) compute $a_0, \ldots, a_5$, (ii) use generating functions to find a formula for a_n in terms of n, and (iii) verify your formula for $n \le 5$.

(a) $a_0 = 1$, $a_1 = 3$, $a_n = 3a_{n-2}$

(b) $a_0 = 3$, $a_1 = 1$, $a_n = 2a_{n-1} + 3a_{n-2}$

(D2) *Power series derivatives.* Fix formal power series $A(z)$ and $B(z)$. The goal of this problem is to prove some of the "derivative rules" from Calculus.

(a) Prove that if $k \ge 1$, then $\frac{d}{dz}[(B(z))^k] = k(B(z))^{k-1}B'(z)$.
Hint: this can be done without writing any sigma-sums, instead using product rule and induction on k.

(b) Prove that if $B(z)$ has constant term **zero**, then $\frac{d}{dz}[A(B(z))] = A'(B(z))B'(z)$.
Hint: use part (a) and sigma-sums, but avoid writing "b_n" at all costs.

(D3) *Generating functions of polynomials.* We saw in class that for each $d \ge 0$, there exists a polynomial $Q_d(z)$ such that

$$\sum_{n \ge 0} n^d z^n = \frac{Q_d(z)}{(1-z)^{d+1}}.$$

(a) Write down $Q_1(z)$, $Q_2(z)$, and $Q_3(z)$ for reference (you may consult your notes).

(b) Using formal derivatives, obtain an expression for $Q_d(z)$ in terms of $Q_{d-1}(z)$.

(c) Use your expression from part (a) to show that $Q_d(z)$ has degree d.

(d) Use your expression from part (a) to show that $Q_d(z)$ has no constant term.

(e) Find a formula for $Q_d(1)$. Use your expression from part (a) to verify this.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use generating functions to find a formula for a_n if $a_0 = 1$ and $a_n = 3a_{n-1} + 2^n$.

(H2) Recall that the *Fibonacci sequence* f_n is defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Use generating functions to show that

$$f_n = \frac{1}{\sqrt{5}}\,\omega^n - \frac{1}{\sqrt{5}}\,\overline{\omega}^n,$$

where

$$\omega = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \overline{\omega} = \frac{1-\sqrt{5}}{2}$$

Hint: begin by verifying that $1 - z - z^2 = (1 - \omega z)(1 - \overline{\omega} z)$.

(H3) The goal of this problem is to prove the *quotient rule* for power series.

(a) Prove that if $B(z)$ has a **nonzero** constant term, then

$$\frac{d}{dz}\left[(B(z))^{-1}\right] = \frac{-B'(z)}{(B(z))^2}.$$

Hint: to save a LOT of algebra, take the derivative of both sides of $B(z)(B(z))^{-1} = 1$ using the product rule.

(b) Prove that if $B(z)$ has a **nonzero** constant term, then

$$\frac{d}{dz}\left[\frac{A(z)}{B(z)}\right] = \frac{A'(z)B(z) - A(z)B'(z)}{(B(z))^2}.$$

Hint: using part (a) and the product rule can save a LOT of algebra.

(H4) Resuming notation from Problem (D3), prove the coefficients of the positive-degree terms of $Q_d(z)$ are positive integers.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove the coefficients of $Q_d(z)$ are symmetric (that is, if we write $Q_d(z) = a_d z^d + \cdots + a_1 z$, then $a_i = a_{d+1-i}$ for each i).