Fall 2021, Math 596: Week 4 Problem Set
Due: Thursday, September 23rd, 2021
Quasipolynomials and Rational Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Eventual polynomials.* Thoughout this problem, suppose $Q(z)$ is a polynomial, $d \geq 0$, and

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{Q(z)}{(1-z)^{d+1}}.$$

We proved in class a_n is a polynomial in n of degree d iff $\deg Q(z) < d+1$ and $Q(1) \neq 0$.

(a) Find a formula for a_n in terms of n if $d = 2$ and $Q(z) = 5 - 6z + z^2$. Is the degree of a_n as expected by the theorem from class?

(b) Find $Q(z)$ when $d = 0$ and $a_n = n$. Why does this not violate the theorem from class?

(c) Find a formula for a_n in terms of n if $d = 2$ and $Q(z) = 1 + z + z^2 + z^3 + z^4$.
Hint: begin by using polynomial long division.

(d) Based on your answer to the previous part, conjecture a mathematical definition of "eventually polynomial" so that the following theorem holds.

Theorem. *If $d \geq 0$, then $Q(z)$ is a polynomial if and only if a_n is eventually polynomial of degree at most d.*

(e) Find d and $Q(z)$ when $a_2 = 17$ and $a_n = n^2$ for $n \neq 2$.

(f) Prove the theorem in part (c).
Hint: use the polynomial long division and the theorem from class.

(D2) *Quasipolynomials.* In this problem, suppose $Q(z)$ is a polynomial, $d \geq 0$, $p \geq 1$, and

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{R(z)}{(1-z^p)^{d+1}}.$$

We saw in class that a_n is a quasipolynomial in n of degree at most d and period dividing p if and only if $\deg R(z) < p(d+1)$.

(a) Find d, p, and $R(z)$ when

$$a_n = \begin{cases} \frac{1}{4}n^2 + n + 1 & \text{if } n \equiv 0 \bmod 2; \\ \frac{1}{4}n^2 + n + \frac{3}{4} & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

(b) Find a formula for a_n if $d = 3$, $p = 2$, and $R(z) = z + 2z^2 + 3z^3 + 4z^4 + 5z^5$.

(c) Prove the theorem at the beginning of this problem.
Hint: start by proving the forward direction, and use the theorem from the beginning of Problem (D1) in your proof. Then, for the backward direction, use linear algebra (and don't hesitate to ask questions!).

(d) Develop (and prove) an analog of the theorem in Problem (D1)(c) for when a_n is "eventually quasipolynomial" in n.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) This problem concerns the theorem you obtained in Problem (D1)(c), with the following definition: we say a_n is *eventually polynomial* if there exists a polynomial $f(n)$ and an integer N such that $a_n = f(n)$ for all $n \geq N$.

Given $Q(z)$ and d, describe a method of determining

(a) the degree of $f(n)$,

(b) the leading coefficient of $f(n)$, and

(c) the smallest integer N such that $a_n = f(n)$ for all $n \geq N$.

Be sure to prove your claim in each part.

(H2) Resume notation from Problem (H1). Prove or disprove: if $Q(z)$ has integer coefficients, then $f(n)$ has integer coefficients. What about the converse?

(H3) Prove that a_n is a quasipolynomial of degree d, period dividing p, and constant leading coefficient if and only if

$$\sum_{n\geq 0} a_n z^n = \frac{Q(z)}{(1-z)(1-z^p)^d}$$

for some polynomial $Q(z)$ with $\deg Q(z) < pd + 1$ and $Q(1) \neq 0$.

(H4) We saw in class that if

$$\sum_{n\geq 0} a_n z^n = \frac{1}{(1-z^3)(1-z^4)(1-z^5)}$$

then a_n equals the number of ways to write n as a sum of 3's, 4's, and 5's (called *restricted partitions* of n). Prove a_n is a quasipolynomial in n with constant leading coefficient. Find its degree, period, and leading coefficient.

Hint: use Problem (H3) and the fact that $1 - z^4$ has (complex) roots 1, -1, i, and $-i$. In particular, do **not** find a formula for a_n (you will see why once you find the period).

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 1}$ and distinct nonzero $r_1, \ldots, r_k \in \mathbb{C}$. Prove that if $Q(z)$ is any polynomial with $\deg Q(z) < m_1 + \cdots + m_k$, we can find constants $c_{j,\ell} \in \mathbb{C}$ such that

$$\frac{Q(z)}{(1-r_1 z)^{m_1} \cdots (1-r_k z)^{m_k}} = \sum_{j=1}^{k} \left(\frac{c_{j,1}}{1-r_j z} + \frac{c_{j,2}}{(1-r_j z)^2} + \cdots + \frac{c_{j,m_j}}{(1-r_j z)^{m_j}} \right).$$

Note: though notationally dense, this is just *partial fraction decomposition* of the left hand side, written out formally.