Fall 2021, Math 596: Week 5 Problem Set
Due: Thursday, September 30th, 2021
The Bridge from Power Series to Geometry

Discussion problems. The problems below should be worked on in class.

(D1) *Power series to geometry.* In this problem, we will explore a geometric interpretation of rational power series.

(a) Using power series multiplication, find all nonzero terms in

$$A(z_1, z_2) = \frac{1}{(1 - z_1^3 z_2)(1 - z_2^2)}$$

with total degree at most 10. Plot their exponents as points in $\mathbb{R}^2$.

(b) Will any of the coefficients of $A(z_1, z_2)$ be larger than 1?

(c) Do the same for the power series

$$B(z_1, z_2) = \frac{1}{(1 - z_1^2)(1 - z_1 z_2)(1 - z_2^2)}.$$

Label each point with its coefficient in $B(z_1, z_2)$.

(d) On the same axes as part (c), label each point with its coefficient in

$$C(z_1, z_2) = \frac{1 - z_1^2 z_2^2}{(1 - z_1^2)(1 - z_1 z_2)(1 - z_2^2)}.$$

(e) Does it appear like any of the terms in $C(z_1, z_2)$ will have coefficient larger than 1? What is the relationship between the point $(2, 2)$ in part (c) and the term "$-z_1^2 z_2^2$" in the numerator of $C(z_1, z_2)$?

(D2) *Geometry to power series.* Find a rational expression for

$$A(z_1, z_2) = \sum_{(a,b) \in S} z_1^a z_2^b$$

for each of the following sets $S \subset \mathbb{Z}^2_{\geq 0}$. You do **not** have to simplify your answer.

(a) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b\}$

(b) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : 2a < b\}$
Hint: use part (a) to your advantage.

(c) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b \text{ and } b \geq 2\}$
Hint: use part (a) to your advantage.

(d) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b \text{ and } 2b \geq a\}$

(e) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : a \geq 1 \text{ and } 1 \leq b \leq 4\}$

(f) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : a \geq 1 \text{ and } 1 \leq b \leq 400\}$
Note: try to do this with as small of a numerator as possible.

(g) $S = \{(a, b) \in \mathbb{Z}^2_{\geq 0} : a + b \equiv 0 \bmod 3\}$

(D3) *A 3D example.* The goal of this problem is to find a rational expression for the power series

$$A(z_1, z_2, z_3) = \sum_{(a,b,c)\in S} z_1^a z_2^b z_3^c$$

where $S = \{(a, b, c) \in \mathbb{Z}^2_{\geq 0} : a \leq c \text{ and } b \leq c\}$.

(a) Find all elements $(a, b, c) \in S$ with $c = 1$. Do the same for $c = 2$.

(b) Use part (a) to (roughly) sketch S. Draw the cross section $c = 1$ in a different color.

(c) On Tuesday in class, we saw that

$$\sum_{(a,b,c)\in T} z_1^a z_2^b z_3^c = \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)},$$

where $T = \{(a, b, c) \in \mathbb{Z}^3_{\geq 0} : a \leq b \leq c\}$. Use this (and symmetry) to find $A(z_1, z_2, z_3)$.

(d) Consolidate your answer to (c) into a single fraction. Interpret it's numerator.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Given points $(a_1, b_1), \ldots, (a_k, b_k) \in \mathbb{Z}_{\geq 0}^2$, the set

$$St((a_1, b_1), \ldots, (a_k, b_k)) = \{(a, b) \in \mathbb{Z}_{\geq 0} : \text{there exists an } i \text{ with } a \geq a_i \text{ and } b \geq b_i\}$$

is called the *staircase* generated by $(a_1, b_1), \ldots, (a_k, b_k)$. Find $Q(z_1, z_2)$ so that

$$\sum_{(a,b) \in S} z_1^a z_2^b = \frac{Q(z_1, z_2)}{(1 - z_1)(1 - z_2)}$$

for each of the following sets $S \subset \mathbb{Z}_{\geq 0}^2$.

(a) $S = St((3, 1), (0, 2))$

(b) $S = St((0, 2), (1, 1), (2, 0))$

(c) $S = St((0, 4), (1, 1), (2, 3), (3, 0))$

(H2) Find a rational expression for the formal power series

$$A(z_1, z_2, z_3) = \sum_{(a,b,c) \in S} z_1^a z_2^b z_3^c$$

for each of the following sets $S \subset \mathbb{Z}_{\geq 0}^3$.

(a) $S = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^3 : 2a \geq b + 1,\ 2b \geq a + 1,\ a + b \geq 3, \text{ and } c = 0\}$

(b) $S = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^3 : a + b \leq 2\}$

(c) $S = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^3 : a + b \geq c,\ b + c \geq a,\ a + c \geq b, \text{ and } a + b \leq 3c\}$
Hint: to help visualize this set, consider the cross sections with $c = 1$ and $c = 2$. Alternatively, find all 5 points of S with last coordinate $c = 1$, then all 13 points with last coordinate $c = 2$.

(H3) Find a rational expression for each of the following.

(a) $A(z_1, z_2) = \sum_{a,b \geq 0} \min(a, b)\, z_1^a z_2^b$

(b) $A(z_1, z_2) = \sum_{a,b \geq 0} \max(a, b)\, z_1^a z_2^b$

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Characterize which functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ satisfy

$$\sum_{n \geq 0} f(n) z^n = \frac{Q(z)}{R(z)}$$

for some polynomials $Q(z)$ and $R(z)$ with coefficients in $\mathbb{C}$ and $R(0) \neq 0$.