Fall 2021, Math 596: Week 6 Problem Set
Due: Thursday, October 7th, 2021
Polytopes, Polyhedra, and Cones

Discussion problems. The problems below should be worked on in class.

(D1) *Warmup.* Consider the polytope $P = H \cap P'$, where

$$H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\} \qquad \text{and} \qquad P' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \le x_i \le 2\}$$

After comparing answers to the preliminary problems, find the H- and V-descriptions of P.

(D2) *The V-description of a pointed cone.* Every pointed cone C can be written as the non-negative span of a finite set of vectors, i.e., $C = \text{span}_{\ge 0}\{v_1, \ldots, v_k\}$. Analogous to how the V-description of a polytope P is the (unique) minimal set of points whose convex hull equals P, the *V-description* of a pointed cone C is the unique minimal set of ray vectors whose non-negative span equals C. One way to think about this: if a cross-section P of a pointed cone C is a polytope, the vertices of P are the points in P that lie on rays of C.

(a) Find the V-description of each of the following pointed cones.

(i) $C = \{(x_1, x_2) \in \mathbb{R}^2 : 3x_1 \le 2x_2,\ 7x_2 \le 5x_1\}$
(ii) $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \le x_1 \le x_2 \le x_3\}$
(iii) $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 \ge x_2,\ 2x_3 \ge x_2,\ x_1 + x_2 \ge x_3,\ x_2 + x_3 \ge x_1\}$
(iv) $C = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 \ge x_3,\ x_2 + x_3 \ge x_4,\ x_3 + x_4 \ge x_1,\ x_4 + x_1 \ge x_2\}$

(b) Find the H-description of each of the following pointed cones.

(i) $C = \text{span}_{\ge 0}\{(1, 2), (2, 1)\}$
(ii) $C = \text{span}_{\ge 0}\{(1, 1, 0), (0, 1, 1), (1, 0, 1), (3, 1, 1), (1, 3, 1), (1, 1, 3)\}$
(iii) $C = \text{span}_{\ge 0}\{(1, 0, 0, 1), (-1, 0, 0, 1), (0, 1, 0, 1), (0, -1, 0, 1), (0, 0, 1, 1), (0, 0, -1, 1)\}$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each of the following polytopes P, find the (irredundant) H-description (in the form $Ax \le b$) and V-description of P.

Hint: begin by drawing P (as best you can).

(a) $P = \text{conv}\{(-1,-1),(1,-1),(-1,0),(0,0),(0,1),(1,1),(2,1)\}$.

(b) $P = \text{conv}\{(1,0,0),\ (0,1,0),\ (-1,-1,0),\ (0,0,1),\ (0,0,-1)\}$.

(H2) Given a polytope $P = \text{conv}\{v_1,\dots,v_k\} \subseteq \mathbb{R}^d$, the *pyramid over* P is the polytope

$$\text{Pyr}(P) = \text{conv}(P \cup \{e_{d+1}\}) \subseteq \mathbb{R}^{d+1}.$$

Assuming P has H-description $Ax \le b$, find the H-description of $\text{Pyr}(P)$.

(H3) Prove from first principles (i.e., by showing set containment both ways) that

$$\text{conv}(\{0,1\}^d) = [0,1]^d.$$

To clarify the notation here, $[0,1]^d$ is the d-dimensional cube, and $\{0,1\}^d$ is the set of d-dimensional 01-vectors (i.e., the vertices of the d-cube).

(H4) The *permutohedron* $P_n \subset \mathbb{R}^n$ is the convex hull of all points whose coordinates are some reordering of $(1,2,\dots,n)$. For example,

$$P_3 = \text{conv}\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Prove that $\dim P_n = n-1$ by doing the following:

(i) locate a single linear equation that is satisfied by every vertex of P_n, and argue that this proves $\dim P_n \le n-1$; and

(ii) locate $n-1$ linearly independent differences $v-w$ of vertices of P_n, and argue that this proves $\dim P_n \ge n-1$.

(H5) Consider the polyhedron

$$R = \{(a,b) \in \mathbb{R}^2 : 2a \ge b+1,\ 2b \ge a+1, \text{ and } a+b \ge 3\}.$$

(a) Locate a polytope P and a cone C such that $R = P + C$, where

$$P + C = \{p + c : p \in P \text{ and } c \in C\}.$$

(b) Is your choice of P unique? Is your choice of C unique?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) The Birkhoff polytope $B_n \subset \mathbb{R}^{n^2}$ has as its vertices the 01-matrices with exactly one 1 in each row and column (there are $n!$ such matrices). Find $\dim B_n$.