Fall 2021, Math 596: Week 7 Problem Set
Due: Thursday, October 14th, 2021
Faces of Polyhedra

Discussion problems. The problems below should be worked on in class.

(D1) *Edges of cubes.*

(a) Draw the cubes $C_2 \subset \mathbb{R}^2$ and $C_3 \subset \mathbb{R}^3$. Label the vertices in each drawing.

(b) Formulate a conjecture on when two vertices v and w of C_d are connected by an edge. The goal of this problem is to prove your conjecture, starting with C_3.

(c) For each edge e connecting vertices v and w of C_3, find an equation of a hyperplane H (which should have the form $a_1x_1 + a_2x_2 + a_3x_3 = b$ for some $a_1, a_2, a_3, b \in \mathbb{R}$) so that (i) the only vertices H contains are v and w, and (ii) the remaining vertices of C_3 lie on the same side of H. This ensures H is "just touching the polytope" at e.

Hint: be systematic, and use symmetry to your advantage!

(d) Select specific vertices v and w of C_3 that are *not* connected by an edge (e.g., opposite sides of a square face). Locate a point that is (i) a convex combination of v and w, and (ii) a convex combination of some other collection of vertices of C_3. Do this with another pair of vertices not connected by an edge (e.g., opposite corners of the cube).

(e) Briefly explain why the previous part proves there is not an edge between v and w.

(f) Generalize your arguments above to obtain a characterization (with proof!) for the edges of the d-dimensional cube C_d for $d \geq 3$.

(D2) *Ridges of cubes.*

(a) Draw a picture of C_3, and, using the picture of C_4 drawn in lecture as a guide, draw C_4. With these as a guide, conjecture which pairs of facets of C_d form a ridge.

(b) For each pair F, F' of facets of C_4, locate $d - 2$ linearly independent vectors that are differences of vertices in $F \cap F'$. Conclude that $F \cap F'$ is a ridge of C_d.

(c) With F and F' as in the previous part, verify $F \cap F'$ is indeed a face by locating a halfspace H such that each vertex of C_d lies in $F \cap F'$ if and only if it lies on the boundary of H.

Note: this is not strictly needed since any intersection of any collection facets is a face.

(d) For each pair of facets of C_d that you conjecture do **not** intersect to form a ridge, prove your claim.

(e) Locate a polytope with two facets F and F' whose intersection is a non-empty face that is not a ridge.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Locate a family of 3-dimensional polytopes that demonstrates #facets - #vertices can be arbitrarily large. Can the same be said for #vertices - #facets?

Note: for this problem, a brief informal description/argument accompanied by two or three drawings is sufficient.

(H2) Prove that any two vertices of the d-simplex

$$S_d = \mathrm{conv}\{0, e_1, \ldots, e_d\}$$

share an edge (here, e_i denotes the i-th standard basis vector).

(H3) Consider the cone

$$C = \mathrm{span}_{\geq 0}\{(3,0,1,0,0), (2,1,1,0,0), (2,0,1,1,0), (1,1,1,1,0), (2,0,1,0,1)\}.$$

Hint: this **can** be visualized faithfully in 3D, after a couple of clever reductions!

(a) Find the H-description of C.

(b) Find a hyperplane defining each ray of C.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) A polytope is called *neighborly* if every pair of vertices is connected by an edge (for example, the simplex in Problem (H2) is neighborly). Prove that the Birkhoff polytope $B_n \subseteq \mathbb{R}^{n^2}$ is neighborly if and only if $n \leq 3$.

Announcement. For those who are want to avoid drawing polytopes by hand, there is a free web app you can use, developed by Nils Olsson (an SDSU student from a prior Math 596).

`https://nilsso.github.io/pages/math/semi-comb/polytope.html`