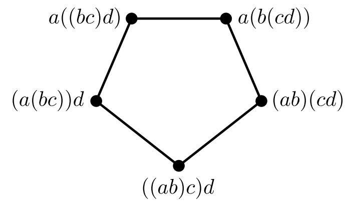
Fall 2021, Math 596: Week 8 Problem Set
Due: Thursday, October 21st, 2021
Face Lattice Combinatorics

Discussion problems. The problems below should be worked on in class.

(D1) *Associahedra.* The goal of this problem is to build and draw a new 3-dimensional polytope.

(a) The *associahedron* A_n is a polytope whose vertices are in bijection with the ways of associating parenthesis when multiplying n elements. The polytope A_4 (dimension 2) is depicted below. What is the "rule" for when two vertices are connected by an edge?



(b) List all associations of $abcde$, e.g. $a(b((cd)e))$ or $(ab)((cd)e)$. There are 14 total.

(c) Draw a graph whose vertices are the 14 expressions you found above, and where an edge is drawn between any two vertices that differ by moving parenthesis exactly one.

(d) Find a way to draw the above graph so that no edges cross.

(e) Using the above graph as a starting place, draw the associahedron A_5. Suggestions:
- Take your time and be methodical in your drawing. Patience is greatly rewarded.
- It may take several revisions to find a "satisfying" drawing, so don't be afraid to experiment and try some different things!
- There is no single "correct" answer here. Unlike most "common" shapes like the cube or octohedron, many different drawings are possible.

(D2) *Permutohedra.* An *ordered set partition of* n is a list $B_1, \ldots, B_r$ of disjoint nonempty sets (called *blocks*) whose union equals $\{1, 2, \ldots, n\}$. As an example,

$$(\{2,4\}, \{1,3,7\}, \{5,6\}) \qquad \text{(shorthand: 24|137|56)}$$

is an ordered set partition of $n = 7$ with $r = 3$ blocks. Within each block the order does not matter (since blocks are sets), but the order of the blocks themselves does.

Recall that the permutohedron $P_n \subseteq \mathbb{R}^n$ is the convex hull of all points whose coordinates are some reordering of $(1, 2, \ldots, n)$. It turns out each nonempty face of P_n corresponds to an ordered set partition of $\{1, 2, \ldots, n\}$. More specifically, if

$$(B_1, \ldots, B_r) \longmapsto F,$$

then a vertex $w = (w_1, \ldots, w_n) \in F$ if and only if $w_i < w_j$ whenever $i \in B_k$ and $j \in B_{k+1}$.

(a) Find all vertices in the face corresponding to each of the following.

$$2|5|1|3|4 \qquad 2|15|3|4 \qquad 124|3 \qquad 24|13|5$$

(b) Draw P_2 and P_3, and next to each face write the corresponding ordered set partition.

(c) Determine when two ordered set partitions correspond to faces F and F' with $F' \subset F$.

(d) Determine how to identify $\dim F$ from the corresponding ordered set partition.

(e) The vertices on each edge of P_3 coincide in one coordinate. Use this to obtain an inequality defining each facet of P_3.
Note: these form the H-description of P_3 along with $x_1 + x_2 + x_3 = 6$.

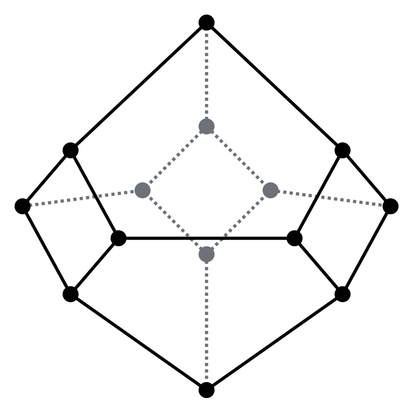
(f) Each facet of P_4 is either a hexagon or a square. What property of the corresponding ordered set partition determines which it is?

(g) Find an inequality defining each facet of P_4. Generalize to P_n.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Below is a picture of the 3-dimensional associahedron A_5, and the final page of this document depicts the face lattice of A_5 (except for the empty face), where each face is represented by a collection of non-crossing diagonals of a hexagon. Your goal for this problem is to connect these depictions and explore the structure of the face lattice. Your answer should be a paragraph or two addressing the following (plus any other patterns you notice).

- When is one collection of non-crossing diagonals "above" another in the face lattice?
- Given a collection of non-crossing diagonals, what determines the dimension of the corresponding face?
- What determines which facets are square vs pentagonal? In higher dimensional associahedra, what will the different "types" of facets be?
- Each row of the face lattice is depicted with certain "clumps" (there are 5 in the bottom row, 5 in the middle row, and 3 in the top row). Why are these clumped as they are? What distinguishes each clump?
- Broadly speaking, how was the ordering (left to right) of faces chosen in each row of the face lattice?



(H2) Prove that any subset of the vertices of the d-simplex

$$S_d = \text{conv}\{0, e_1, \ldots, e_d\}$$

is the V-description of a face of S_d (here, e_i denotes the i-th standard basis vector).

(H3) Find a formula for the number of ridges of the d-dimensional cube C_d.

(H4) Complete and write up the last part of Problem (D2).

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a halfspace defining each face of the permutohedron P_n.