Fall 2021, Math 596: Week 9 Problem Set
Due: Thursday, October 28th, 2021
Ehrhart Functions and Ehrhart Series

Discussion problems. The problems below should be worked on in class.

(D1) *Computing Ehrhart series.* Recall from lecture that if $P \subset \mathbb{R}^d$ is a polytope, then

$$\mathrm{Ehr}_P(z) = \sum_{n \geq 0} L_P(n) z^n$$

is the *Ehrhart series* of P.

(a) The *cone over* P, denoted $\mathrm{cone}(P)$, is the cone $C = \mathrm{span}_{\geq 0}(P \times \{1\}) \subset \mathbb{R}^{d+1}$. Identify which polytope the cone in Problem (P1) is the cone over.

(b) Express $\mathrm{Ehr}_P(z)$ in terms of $\sigma_C(z_1, \ldots, z_{d+1})$.

(c) Fix integers $a < b$, and let $P = [a, b] \subset \mathbb{R}$, that is, the closed interval from a to b (this is a 1-dimensional polytope with vertices a and b). **Using part (b)**, find the Ehrhart series of P in terms of a and b.

(d) Let $P = \mathrm{conv}\{(0, 2), (1, 0), (1, 1)\}$. Draw P, $2P$, and $3P$ on **different** sets of axes. Also draw the cross sections $x_3 = 1$, $x_3 = 2$, and $x_3 = 3$, respectively, of the fundamental parallelopiped of $C = \mathrm{cone}(P)$ (use a different color).

(e) Find $\mathrm{Ehr}_P(z)$ for the polytope P in the previous part **using part (b)**.

(f) Find the Ehrhart series of the pentagon

$$P = \mathrm{conv}\{(-1, -1), (1, -1), (-1, 0), (1, 0), (0, 1)\}$$

by triangulating and then using part (b). Use this to find a formula for the Ehrhart polynomial $L_P(t)$.

(g) Consider the rational polygon $P = \mathrm{conv}\{(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2})\}$. **Using part (b)**, show

$$\mathrm{Ehr}_P(z) = \frac{1 + z}{(1 - z^2)^3}.$$

Use this to find a formula for $L_P(t)$ (remember, this will be a *quasipolynomial*).

(D2) *Ehrhart series of the cross polytope.* Given a polytope $P = \mathrm{conv}\{v_1, \ldots, v_k\} \subset \mathbb{R}^d$, define

$$\mathrm{Pyr}(P) = \mathrm{conv}\{(v_1, 0), (v_2, 0), \ldots, (v_k, 0), e_{d+1}\} \subset \mathbb{R}^{d+1} \quad \text{and}$$
$$\mathrm{Bipyr}(P) = \mathrm{conv}\{(v_1, 0), (v_2, 0), \ldots, (v_k, 0), e_{d+1}, -e_{d+1}\} \subset \mathbb{R}^{d+1},$$

the *pyramid* and *bipyramid* over P, respectively (each lives in one dimension more than P).

(a) Let $P = [-1, 1] \subset \mathbb{R}$. Draw P, $\mathrm{Pyr}(P)$ and $\mathrm{Bipyr}(P)$.

(b) Let $P = \mathrm{conv}\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Draw P, $\mathrm{Pyr}(P)$ and $\mathrm{Bipyr}(P)$.

(c) Find a formula relating $L_{\mathrm{Pyr}(P)}(3)$ in terms of $L_P(1)$, $L_P(2)$, and $L_P(3)$.

(d) Let $A(z) = \sum_{n \geq 0} a_n z^n$ and $B(z) = \sum_{n \geq 0} b_n z^n$, and suppose each $b_n = a_0 + \cdots + a_n$. Find an equation relating $A(z)$ and $B(z)$ that uses no sigma-sums.

(e) Find a formula for $\mathrm{Ehr}_{\mathrm{Pyr}(P)}(z)$ in terms of $\mathrm{Ehr}_P(z)$ (again involving no sigma-sums).

(f) Find a sigma-sum free equation relating the Ehrhart series of P, $\mathrm{Pyr}(P)$, and $\mathrm{Bipyr}(P)$. Use this to obtain a formula for $\mathrm{Ehr}_{\mathrm{Bipyr}(P)}(z)$ in terms of $\mathrm{Ehr}_P(z)$.

(g) Find a formula for the Ehrhart series of the d-dimensional cross polytope

$$P_d = \mathrm{conv}\{e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d\} \subset \mathbb{R}^d.$$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the Ehrhart series of the 2-dimensional permutohedron

$$P_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Hint: locate a projection of P_3 into $\mathbb{R}^2$ that preserves the Ehrhart function, then triangulate.

(H2) Suppose $h \in \mathbb{Z}_{\geq 1}$, and consider

$$T = \{(0,0,0),(1,0,0),(0,1,0),(1,1,h)\},$$

known as *Reeve's tetrahedron*.

(a) Find $\mathrm{Ehr}_T(z)$, the Ehrhart series of Reeve's tetrahedron (your answer will have h's).

(b) Use part (a) to find a formula for $L_T(t)$, the Ehrhart function of T.

(H3) Fix two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, and define

$$P \times Q = \{(p_1,\ldots,p_n,q_1,\ldots,q_m) : (p_1,\ldots,p_n) \in P, (q_1,\ldots,q_m) \in Q\} \subset \mathbb{R}^{n+m}.$$

(a) Prove $P \times Q$ is a polytope.

(b) Express $L_{P\times Q}(t)$ in terms of $L_P(t)$ and $L_Q(t)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $a, b \in \mathbb{Z}_{\geq 1}$ with $\gcd(a,b) = 1$, and consider the rhombus

$$P = \{(x,y) : a|x| + b|y| \leq ab\}.$$

Find $L_P(t)$ in terms of a, b, and t.