Fall 2021, Math 596: Week 10 Problem Set
Due: Thursday, November 4th, 2021
Triangulations

Discussion problems. The problems below should be worked on in class.

(D1) *Triangulations.* Recall a *triangulation* of a polytope P is a collection $\mathcal{T}$ of simplices whose union equals P and for which any pairwise intersection of simplices is a face of both.

(a) Locate a triangulation of $P = \text{conv}\{e_1, -e_1, e_2, -e_2\}$ using the vertices of P and the origin. Then locate a triangulation that only uses the vertices of P.

(b) Locate a triangulation of the octohedron with 8 simplices and 1 “new” vertex.

(c) Locate a triangluation of the octohedron with no “new” vertices. Attempt to use the smallest possible number of simplices.

(d) Locate a triangulation of the d-dimensional cross polytope.

(e) The polytope $P = \text{conv}\{(1, 0, 0), (0, 1, 0), (-1, -1, 0), (0, 0, 1), (0, 0, -1)\}$ has exactly 2 triangulations that use no “new” vertices. Find both.

(f) Let $C_3 = [0, 1]^3$ denote the 3-cube. Argue that

$$S = \{p \in C_3 : p_1 \leq p_2 \leq p_3\}$$

is a simplex. Use this (and symmetry) to locate a triangulation of C_3 with 6 simplices.

(g) Locate a triangulation of C_3 with strictly less simplices than the previous part.

(h) Locate a triangulation of the d-dimensional cube $[0, 1]^d$.

(i) Locate a triangulation of the d-dimensional cross polytope that uses no “new” vertices. Hint: this can be done with 2^{d-1} simplices.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Given a polytope $P = \text{conv}\{v_1, \ldots, v_k\} \subset \mathbb{R}^d$, define

$$\text{Pyr}(P) = \text{conv}\{(v_1, 0), (v_2, 0), \ldots, (v_k, 0), e_{d+1}\} \subset \mathbb{R}^{d+1},$$

the *pyramid* over P (this lives in one dimension more than P).

(a) Given a triangulation $\mathcal{T}$ of a polytope $P \subset \mathbb{R}^d$, conjecture a triangulation of $\text{Pyr}(P)$ with the same number of simplices as $\mathcal{T}$. State your claim rigorously!

(b) Prove your conjecture in part (a) **from first principles** (i.e., use the definition of convex hull, and prove any set equality statements by arguing containment both ways).

Hint: first prove, using the definition of convex hull, that every $p \in \text{Pyr}(P)$ with $p \neq e_{d+1}$ lies on a line segment connecting e_{d+1} to a point $p' \in P \times \{0\}$.

(H2) One method of constructing a triangulation $\mathcal{T}$ for a general polytope P is via the following inductive process:

(i) find a triangulation $\mathcal{T}_F$ each facet F of P;

(ii) choose a point q in the interior of P; and

(iii) for each facet F of P and each simplex $\text{conv}\{v_1, \ldots, v_d\} \in \mathcal{T}_F$, include a simplex $\text{conv}\{v_1, \ldots, v_d, q\}$ in $\mathcal{T}$.

Using this process inductively to every positive-dimensional face of P (i.e., splitting each edge in half, then triangulating each 2-dimensional face using one "new" vertex, etc., eventually building up to a triangulation of P itself) is known as *barycentric subdivision.*

Find the barycentric subdivision of each of the following polytopes.

(a) The pentagon $P = \text{conv}\{(0, 2), (2, 0), (2, -2), (-2, -2), (-2, 0)\}$ (your triangulation should have 10 simplices).

(b) The octohedron $P = \text{conv}\{6e_1, -6e_1, 6e_2, -6e_2, 6e_3, -6e_3\}$.

(H3) Determine the number of simplices in the barycentric subdivision of the d-dimensional cube $C_d = [0, 1]^d$.

(H4) Sketch a proof that the barycentric subdivision of any polytope P is a triangulation of P. You may assume $P \subset \mathbb{R}^d$ is full dimensional.

Hint: proceed by induction on d, and use the fact that every point $p \in P \setminus \{q\}$ lies on a line segment connecting q and some point p' on the boundary of P.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Appendix A of *Computing the Continuous Discretely* has a proof that every polytope P has a triangluation using no "new" vertices (that is, the vertex set of each simplex is contained in the vertex set of P). Read the proof, and summarize the key ideas in the argument.